Т. 9 • № 3 • 2021 ISSN 2308-944X (Print) ISSN 2311-0279 (Online)

Review of Business and Economics Studies

Вестник исследований бизнеса и экономики

DOI: 10.26794/2308-944X

The journal was reregistered in the Federal Service for Supervision of Communications, Informational Technologies, and Mass Media: PI No. ФС77–67072 of 15 September 2016

Издание перерегистрировано в Федеральной службе по надзору в сфере связи, информационных технологий и массовых коммуникаций: ПИ № ФС77–67072 от 15 сентября 2016 г.

Publication frequency – 4 issues per year

Периодичность издания – 4 номера в год

Founder: "Financial University"

Учредитель: «Финансовый университет»

The Journal is included in the core of the Russian Science Citation Index (RSCI)

Журнал включен в ядро Российского индекса научного цитирования (РИНЦ)

The Journal is distributed by subscription. Subscription index: 42137 in the consolidated catalogue "The Press of Russia"

Журнал распространяется по подписке. Подписной индекс 42137 в объединенном каталоге «Пресса России»

Vol. 9 • No. 3 • 2021

Review of Business and Economics Studies

DOI: 10.26794/2308-944X

Review of Business and Economics Studies

REVIEW OF BUSINESS AND ECONOMICS STUDIES (ROBES) is the quarterly peer-reviewed scholarly journal published by the Financial University under the Government of Russian Federation, Moscow. Journal's mission is to provide scientific perspective on wide range of topical economic and business subjects.

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ISSN 2308-944X

Вестник исследований бизнеса и экономики

Редакция научных журналов Финансового университета
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Тел. +7 (499) 553-10-74 (вн. 10-88)
Интернет: https://rbes.fa.ru/jour

Журнал “Review of Business and Economics Studies” («Вестник исследований бизнеса и экономики») зарегистрирован в Федеральной службе по надзору в сфере связи, информационных технологий и массовых коммуникаций 15 сентября 2016 г. Свидетельство о регистрации ПИ № ФС77-67072.

Подписано в печать: 15.07.2021.
Формат 60 × 84 1/8.
Заказ № 644
Отпечатано в Отделе полиграфии Финуниверситета (Ленинградский проспект, д. 49).
16+

Review of Business and Economics Studies
Volume 9, Number 3, 2021

Вестник исследований бизнеса и экономики

№ 3, 2021

Modelling of Financial Markets and Risk-Management

This special issue of the journal contains several papers discovering different aspects of financial markets modelling, option pricing and quantitative risk-management.

The paper by Campoliety, Kato, and Makarov studied a new pricing model with stochastic/randomized volatilities. The models considered here assume that the underlying asset price distributions admit fat tails, which is an attractive feature of the paper. The authors exploit a randomized procedure on the volatility of the Geometrical Brownian Motion model to construct new pricing models developed in detail for the randomized gamma and randomized inverse gamma cases. Both models are characterized by shape and scale parameters and admit closed form analytical density expressions allowing non-arbitrage option prices. The authors have shown that the randomized gamma and inverse gamma models are accurately calibrated to market equity option data.

The paper by *Kozlov and Noga* proposes a methodology for assessing the risk associated with subjective factors that may affect a business project in the context of its information security. The technique developed by the authors uses the fuzzy logic method, which allows determining the dependence of the risks affecting the achievement of the goal of such a business project. The proposed methodology helps avoid incorrect management decisions in the sense of the cost of the project and the effectiveness of the company's personnel policy.

In the paper by *Maximov and Melnikov,* the authors investigated the CVaR methodology of risk-management for spread options. Besides pure theoretical results, an approximative method to determine CVaR is systematically developed. They have shown that the approach works very well in comparison with other methods exploited in this area. Moreover, the paper demonstrates interesting applications to the field of regulatory capital towards the Basel Committee recommendations.

The paper by *Vasilev and Melnikov* is devoted to the method of completions of financial markets. The leading and promising idea of such a method is to replace the set of risk-neutral measures with the equivalent set of completions of the incomplete market under considerations. The paper provides an exposition of this approach which may lead to the dual theory of option pricing in incomplete markets and the markets with different constraints. The paper shows the potential of this method in solving risk-management problems in the context of optimal investment and partial hedging.

In the paper by *Smirnov and Sotnikov*, the authors compared option process in the Bachelier model and the Black-Scholes model with the help of the probability metrics technique. They showed that it is necessary to use different metrics for different options. The authors also demonstrate how to calibrate such metrics by giving illustrative examples.

Dr Alexander Melnikov

ORIGINAL PAPER

DOI: 10.26794/2308-944X-2021-9-3-7-26
JEL C63, G13, G17

Option Pricing under Randomised GBM Models

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ABSTRACT

By employing a randomisation procedure on the variance parameter of the standard geometric Brownian motion (GBM) model, we construct new families of analytically tractable asset pricing models. In particular, we develop two explicit families of processes that are respectively referred to as the randomised gamma (G) and randomised inverse gamma (IG) models, both characterised by a shape and scale parameter. Both models admit relatively simple closed-form analytical expressions for the transition density and the no-arbitrage prices of standard European-style options whose Black-Scholes implied volatilities exhibit symmetric smiles in the log-forward moneyness. Surprisingly, for integer-valued shape parameter and arbitrary positive real scale parameter, the analytical option pricing formulas involve only elementary functions and are even more straightforward than the standard (constant volatility) Black-Scholes (GBM) pricing formulas. Moreover, we show some interesting characteristics of the risk-neutral transition densities of the randomised G and IG models, both exhibiting fat tails. In fact, the randomised IG density only has finite moments of the order less than or equal to one. In contrast, the randomised G density has a finite first moment with finite higher moments depending on the time-to-maturity and its scale parameter. We show how the randomised G and IG models are efficiently and accurately calibrated to market equity option data, having pronounced implied volatility smiles across several strikes and maturities. We also calibrate the same option data to the well-known SABR (Stochastic Alpha Beta Rho) model.
Keywords: static randomisation; pricing European-style options; Black-Scholes implied volatility; calibration; randomised GBM models; SABR model

For citation: **Campolieti, G., Kato, H., & Makarov, R. (2021). Option pricing under randomised GBM models. *Review of Business and Economics Studies, 9*(3),7-26. DOI: 10.26794/2308-944X-2021-9-3-7-26**

ОРИГИНАЛЬНАЯ СТАТЬЯ

Оценка стоимости опционов для рандомизированных моделей геометрического броуновского движения

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АННОТАЦИЯ

Используя процедуру рандомизации дисперсии стандартной модели геометрического Броунского движения (ГБД), авторы построили новые семейства аналитически решаемых моделей ценообразования финансовых активов. В частности, были разработаны два семейства процессов, а именно модели — рандомизированная гамма (Г) и рандомизированная обратная гамма (ОГ), которые характеризуются параметрами формы и масштаба. Обе модели допускают довольно простые аналитические выражения для плотности перехода и безарбитражной цены стандартных европейских опционов. Волатильность Блэка-Шоулза проявляет симметричную «улыбку» для логарифмически форвардной денежности. Примечательно, но для целых значений параметра формы и произвольного положительного параметра масштаба аналитические формулы ценообразования вариантов включают только элементарные функции и даже являются проще стандартных (для постоянной волатильности) формул ценообразования Блэка-Шоулза (модель ГБД). В статье даны характеристики риск-нейтральной плотностей перехода для рандомизированных моделей Г и ОГ, которые демонстрируют «тяжелые хвосты». Рандомизированные плотности для модели ОГ имеют только конечные моменты порядка меньше или равные одному, в то время как рандомизированная плотность для модели Г имеет конечный первый момент и конечные моменты более высокого порядка в зависимости от срока погашения опциона и параметра масштаба. Показано, как рандомизированные модели Г и ОГ могут быть эффективно и точно откалиброваны для рыночных значений опционов, демонстрирующих «улыбку» волатильности для различных цен исполнения и сроков погашения. Откалибровка проведена с помощью модели SABR (Stochastic Alpha Beta Rho). Проведено сравнение этих моделей.

Ключевые слова: статическая рандомизация; ценообразование опционов европейского стиля; подразумеваемая волатильность Блэка-Шоулза; калибровка; рандомизированные модели GBM; модель SABR

1 Introduction

Mathematicians have developed stochastic models to value options. The geometric Brownian motion (GBM) model is known as one of the simplest continuous-time models that admit analytical closed-form formulas for pricing various options (Black & Scholes, 1973). The GBM model is a complete market model where risks can be perfectly hedged. A significant limitation is that there is a discrepancy between anticipated Black-Scholes (BS) prices and the market option prices since the model fails to capture price movements for extreme events (MacBeth & Merville, 1979). Local volatility diffusion models (also known as state-dependent volatility models) are more flexible continuous-time models known for describing the behaviour of implied volatility smile and skew patterns observed in a marketplace. Local volatility diffusion models are also complete market models like the GBM model. In fact, the (one-dimensional) GBM model is simply a local volatility model with constant local volatility.

In some cases, nonlinear local volatility models admit closed-form formulas for pricing various options. Families of local volatility diffusion models that can be analytically solved in closed form have been developed in several papers, see, e.g., Albanese, Campolieti, et al. (2001) and Campolieti and Makarov (2012). They are obtained by applying the "diffusion canonical transformation" to solvable underlying diffusions such as the Bessel, Cox–Ingersoll–Ross and OrnsteinUhlenbeck processes. These models have been shown to calibrate quite well to equity and FX options. One drawback of local volatility diffusion models is the inherent perfect correlation between the underlying asset price and the volatility. In some cases, this contradicts the empirical evidence that they should have an imperfect negative correlation (Rubinstein, 1985).

The stock market is incomplete in many situations as traders cannot use options for hedging all the risks. Stochastic volatility models are incomplete and assume that volatility is a random process. We can make the movements of the underlying asset price and the volatility to be negatively correlated. A first example is the Hull and White stochastic volatility model. Hull and White (1987) derived the closed-form pricing formulas for European vanilla options under their model with zero correlation. They are obtained by averaging the BS prices over the integrated squared instantaneous volatility process. Theoretical results of implied volatility under the GBM model with stochastic volatility are given in Renault & Touzi's paper (1996). They have shown that an implied volatility surface is an even function of the log-forward moneyness and necessarily produces a smile effect under the models with zero correlation. Thus, these models may be used to calibrate to option price market data.

A second example is the Heston model. Heston (1993) successfully applied the Fourier transform method to evaluate European vanilla options with an arbitrary correlation between the asset price and the volatility. He also showed that the distribution of asset returns is asymmetric. Also, he found that when the marginal distributions of the asset returns and the volatility are negatively skewed. Moreover, the BS out-of-the-money (OTM) option prices are negatively biased (i.e., BS OTM option prices are usually smaller when compared to market prices). BS in-the-money (ITM) option prices are positively biased.

A third example is the SABR model introduced by Hagan et al. (2002). The implied volatility curve captured by the SABR model gives consistency with the observed marketplace in dynamics. Other examples are regime-switching models. Bollen demonstrated that the model with two regimes could produce pronounced symmetric smiles in the log-forward moneyness, giving consistency with the higher BS pricing errors for shorter maturities (Bollen, 1998).

This paper constructs new pricing models with randomised volatility, where underlying asset price distributions exhibit fat tails and admit simple closed-form analytical expressions for standard European-style option prices. In particular, we assume that: 1) a unique risk-neutral pricing measure exists (in advance), 2) the underlying asset price processes have a finite first moment but possibly infinite higher moments, 3) there are no correlations between the asset prices and their volatility, and 4) the volatility (squared volatility) coefficient is a random variable with known probability density function (PDF). The assumptions 1) and 2) are based on the Put-Call Parity methodology in Taleb (2015). This methodology neglects the strong (but surreal) assumptions from the dynamic hedging argument and exhibits better practical phenomena in financial markets. The assumptions 1)-4) allow for deriving closed-form expressions (under our new pricing models) by taking a mathematics expectation under diffusion models over the underlying probability distribution for the volatility. Our methodology for computing option prices is closely related to the Bayesian framework in the GBM model studied by Darsinos and Satchell (2007). They considered randomising the volatility where the variance follows the inverse gamma distribution. They were successful in deriving analytically closed-form expressions for the joint PDF of the asset price and the volatility, as well as the marginal PDF of the asset price. However, they could not determine the call pricing formulas analytically, and the option prices could only be obtained numerically.

This paper is organised as follows. Section 2 proposes a general theory of static randomisation under the GBM model, including the almost everywhere (a.e.) existence of transition probability density functions (PDF) of newly constructed asset price processes. We then derive the transition PDFs of the asset price process with static randomisation of the parameter under two families of static randomisation, namely the gamma (G) and the inverse gamma (IG) randomisation. Section 3 states the main results of this paper, including the closed-form expressions of a European vanilla call option and the characteristics of shapes of the implied volatility. In Section 4, we conduct our numerical experiments pertaining to model calibrations to market option data. Finally, we state some concluding remarks with some discussions of future applications.

2 Randomised GBM Models and their Characteristics

Let $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}}, \{\mathcal{F}_t\}_{t\geq 0})$ be some fixed filtered (risk-neutral) probability space where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by the $\widetilde{\mathbb{P}}$-BM. Assume a two-asset economy where the risky asset price (diffusion) process $\{S_t\}_{t\geq 0}$ follows a GBM with stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = r dt + \sqrt{v} d\widetilde{W}_t; \quad S_0 > 0,$$

where r is the constant risk-free rate, v is a constant variance and $\{\widetilde{W}_t\}_{t\geq 0}$ is a standard $\widetilde{\mathbb{P}}$-BM (i.e., Brownian motion under the risk-neutral measure with a bank account as numéraire). The (risk-neutral) transition PDF for this process (for a given variance v) is time-homogeneous, depending on the time difference $\tau \equiv T - t$:

$$\begin{aligned} &\widetilde{\mathbb{P}}_{t,S}(S_T \in dy) \equiv \widetilde{\mathbb{P}}(S_T \in dy \mid S_t = S) = \\ &= \frac{1}{y\sqrt{2\pi v\tau}} e^{-(x+\frac{1}{2}v\tau)^2/2v\tau} dy; \quad S, y > 0, \tau > 0, \end{aligned} \tag{1}$$

where $x = \ln(y/S) - r\tau$. We now consider randomising the parameter v by introducing the random variable $\mathcal{V}$ to distinguish it from the parameter v. Then, we can formulate the pricing function for a standard European-style option with payoff function Λ by:

$$V_{\mathcal{V}}(\tau, S) = e^{-r\tau} \int_{\Omega_{\mathcal{V}}} \widetilde{\mathbb{E}}_{t,S}[\Lambda(S_T)] \mu_{\mathcal{V}}(dv), \quad \tau = T - t. \tag{2}$$

Note that $V_{\mathcal{V}}$ denotes the pricing function for a given choice of the random variable $\mathcal{V}$ on a sample space $\Omega_{\mathcal{V}} \subset \mathbb{R}_+$, where $\mu_{\mathcal{V}}$ is a probability measure for $\mathcal{V}$.[1] In the case of an absolutely continuous random variable $\mathcal{V}$ we have $\mu_{\mathcal{V}}(dv) = \mu_{\mathcal{V}}(v)dv$ with PDF $\mu_{\mathcal{V}}(v)$. The (marginal) transition PDF for the asset price process with randomised volatility **(the randomised GBM process)**, denoted by $\{S_t^{\mathcal{V}}\}_{t\geq 0}$ is defined for fixed $\tau, S > 0$ as:[2]

$$\widetilde{\mathbb{P}}_{t,S}(S_T^{\mathcal{V}} \in dy) \equiv \int_{\Omega_{\mathcal{V}}} \widetilde{\mathbb{P}}_{t,S}(S_T \in dy)\mu_{\mathcal{V}}(dv) \tag{3}$$

We can easily show that the transition PDF integrates to one:

$$\int_0^{\infty} \widetilde{\mathbb{P}}_{t,S}(S_T^{\mathcal{V}} \in dy) = \int_{\Omega_{\mathcal{V}}} \left(\int_0^{\infty} \widetilde{\mathbb{P}}_{t,S}(S_T \in dy) \right) \mu_{\mathcal{V}}(dv) = 1.$$

By a simple application of Fubini's theorem, the transition PDF for $\{S_t^{\mathcal{V}}\}_{t\geq 0}$ is well-defined (a.e.) for every fixed $\tau, S > 0$. We can easily show that the discounted randomised process $\{e^{-rt} S_t^{\mathcal{V}}\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-martingale process [3]

$$\widetilde{\mathbb{E}}_{t,S}\left[S_T^{\mathcal{V}}\right] \equiv \widetilde{\mathbb{E}}\left[S_T^{\mathcal{V}} \mid S_t^{\mathcal{V}} = S\right] = Se^{r\tau}.$$

In what follows, we specify the distribution of $\mathcal{V}$ in two separate ways: as a gamma random variable and as an inverse gamma random variable.

We now look at the transition PDF for the randomised asset price process under the gamma randomisation **(the randomised G process)**, denoted by $\{S_t^{G(\theta,\lambda)}\}_{t\geq 0}$, where $\mathcal{V}$ follows the gamma distribution with shape parameter θ and scale parameter λ (i.e., $\mathcal{V} \sim G(\theta, \lambda)$). The PDF of $\mathcal{V}$ is

$$\mu_{G(\theta,\lambda)}(dv) = \frac{1}{\lambda^{\theta}\Gamma(\theta)} v^{\theta-1} e^{-v/\lambda} dv; \quad \theta, \lambda > 0, \quad \Omega_{G(\theta,\lambda)} = \mathbb{R}_+,$$

where $\Gamma(\theta) = \int_0^{\infty} t^{\theta-1} e^{-t} dt$ is the gamma function. We state a useful integral formula (see Prudnikov, Brychkov, & Marichev, 1986, Eq. 2.3.16.1):

$$\int_0^{\infty} v^{r-1} e^{-pv-q/v} dv = 2\left(\frac{q}{p}\right)^{r/2} \mathrm{K}_r\left(2\sqrt{pq}\right); \quad r \in \mathbb{R}, \quad p, q > 0, \tag{4}$$

where K_ν is the modified Bessel function of the second kind of order ν. It gives the analytical expression for the transition PDF for $\{S_t^{G(\theta,\lambda)}\}_{t\geq 0}$:

$$\widetilde{\mathbb{P}}_{t,S}(S_T^{G(\theta,\lambda)} \in dy) = \frac{e^{-x/2}}{y\sqrt{\pi}\Gamma(\theta)} \left(\frac{2}{\lambda\tau}\right)^{\theta} \left(\frac{\lambda\tau x^2}{8+\lambda\tau}\right)^{\theta/2-1/4} \mathrm{K}_{\theta-1/2}\left(\frac{|x|}{2}\frac{\sqrt{8+\lambda\tau}}{\sqrt{\lambda\tau}}\right) dy.$$

---

[1] One may think that $\mathcal{V}$ is a random variable on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ where $\Omega_{\mathcal{V}}$ is the range of $\mathcal{V}$ and $\mu_{\mathcal{V}}$ is the distribution measure.

[2] Note that $\widetilde{\mathbb{P}}_{t,S}\left(S_T^{\mathcal{V}} \in dy\right) \equiv \widetilde{p}_{\mathcal{V}}(\tau; S, y)dy$, where $\tau = T - t$ and $\widetilde{p}(\tau; S, y) \equiv \widetilde{p}(\tau; S, y \mid v)$ is the transition PDF (in (1)) of the GBM process for a given volatility parameter value v. Hence,

$\dfrac{\widetilde{\mathbb{P}}_{t,s}(S_T^{\mathcal{V}} \in dy)}{dy} \equiv \widetilde{p}_{\mathcal{V}}(\tau; S, y) = \int_{\Omega_{\mathcal{V}}} \widetilde{p}(\tau; S, y \mid v)\mu_{\mathcal{V}}(dv)$ is the transition PDF of the randomized asset price process $\{S_t^{\mathcal{V}}\}_{t\geq 0}$.

[3] Recall that the discounted asset price process under the GBM model $\{e^{-rt} S_t\}_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$-martingale process. Here an underlying filtered probability space for the randomized process is $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}}, \{\mathcal{F}_t^{\mathcal{V}}\}_{t\geq 0})$ where $\mathcal{F}_t^{\mathcal{V}} \equiv \sigma(S_u^{\mathcal{V}}, 0 \leq u \leq t)$.

Where $x = \ln(y/S) - r\tau$. Note that for $\theta = n \in \mathbb{N}$, the transition PDF can be represented by elementary functions. The asymptotic behaviours of the transition PDF at the endpoints are:

$$\frac{\widetilde{\mathbb{P}}_{t,S}(S_T^{G(\theta,\lambda)} \in dy)}{dy} \sim \begin{cases} y^{-3/2+\sqrt{(8+\lambda\tau)/4\lambda\tau}} \left(\ln(1/y)\right)^{\theta-1} & \text{as} \quad y \to 0, \\ y^{-3/2-\sqrt{(8+\lambda\tau)/4\lambda\tau}} (\ln y)^{\theta-1} & \text{as} \quad y \to \infty. \end{cases}$$

Based on these asymptotic expressions, we conclude that the α-moment of the randomised G process:

$$\widetilde{\mathbb{E}}_{t,S}\left[\left(S_T^{G(\theta,\lambda)}\right)^{\alpha}\right] \equiv \int_0^{\infty} y^{\alpha}\, \widetilde{\mathbb{P}}_{t,S}(S_T^{G(\theta,\lambda)} \in dy) < \infty \quad \text{iff} \quad \left|\alpha - \frac{1}{2}\right| < \sqrt{\frac{8+\lambda\tau}{4\lambda\tau}}.$$

It implies the first moment exists, but the second moment exists iff $\lambda\tau < 1$. Furthermore, we have an explicit formula for the second moment:

$$\widetilde{\mathbb{E}}_{t,S}\left[\left(S_T^{G(\theta,\lambda)}\right)^{2}\right] = S^2 e^{2r\tau}(1-\lambda\tau)^{-\theta}; \quad \text{for} \quad \lambda\tau < 1.$$

Let us now consider the transition PDF for the asset price process under the inverse gamma randomisation **(the randomised IG process)**, denoted by $\{S_t^{IG(\theta,\lambda)}\}_{t\geq 0}$. Assume that $\mathcal{V}$ follows the inverse gamma distribution with shape parameter θ and scale parameter λ (i.e., $\mathcal{V} \sim IG(\theta,\lambda)$). The PDF of $\mathcal{V}$ is

$$\mu_{IG(\theta,\lambda)}(dv) = \frac{\lambda^{\theta}}{\Gamma(\theta)}\left(\frac{1}{v}\right)^{\theta+1} e^{-\lambda/v} dv; \quad \theta,\lambda > 0, \quad \Omega_{IG(\theta,\lambda)} = \mathbb{R}_+.$$

By using the integral identity in (4) we obtain the transition PDF for $\{S_t^{IG(\theta,\lambda)}\}_{t\geq 0}$:

$$\widetilde{\mathbb{P}}_{t,S}(S_T^{IG(\theta,\lambda)} \in dy) = \frac{e^{-x/2}}{y\sqrt{\pi}\Gamma(\theta)}\left(\frac{\lambda\tau}{2}\right)^{\theta}\left(x^2 + 2\lambda\tau\right)^{-\theta/2-1/4} \mathrm{K}_{\theta+1/2}\left(\frac{\sqrt{x^2+2\lambda\tau}}{2}\right) dy,$$

where $x = \ln(y/S) - r\tau$. The asymptotics of the transition PDF are now as follows:

$$\frac{\widetilde{\mathbb{P}}_{t,S}(S_T^{IG(\theta,\lambda)} \in dy)}{dy} \sim \begin{cases} y^{-1}\left(\ln(1/y)\right)^{-\theta-1} & \text{as} \quad y \to 0, \\ y^{-2}(\ln y)^{-\theta-1} & \text{as} \quad y \to \infty. \end{cases}$$

These two asymptotics give

$$\widetilde{\mathbb{E}}_{t,S}\left[\left(S_T^{G(\theta,\lambda)}\right)^{\alpha}\right] < \infty \quad \text{iff} \quad 0 \leq \alpha \leq 1.$$

We can see from Figures 1 and 2 that the GBM has the thinnest tail among the three models for $\theta = 1, 2$. The left plot in Figure 1 shows that for $\theta = 1$, the randomised G process has a thinner tail than the randomised IG process for $\theta = 1$. The randomised G process appears to have the thickest tail among the three when $\theta = 2$, but eventually, the randomised G process tails off faster than the randomised IG process, as shown at the right plot in Figure 2. It is interesting to see that the PDF of the randomised G process is uniform for $y \leq Se^{r\tau}$ at the right plot in Figure 1. We can also observe that the PDF of the randomised G process is not differentiable at $y = Se^{r\tau}$ since $\mathrm{K}_{\nu}(z)$ is not differentiable at $z = 0$.

3 Main Results

The conditional risk-neutral probability that the randomised asset price process is above the strike K at a time T can be written as elementary analytical functions for $\theta = n \in \mathbb{N}$.[4] The reader may refer to Appendix 6 for the details. It helps us obtain analytical pricing formulas for European vanilla options. We will illustrate it in this section. The price of a European vanilla call option, denoted by $C_{\mathcal{V}}(\tau, S; K, r)$, can be written in terms of $\widetilde{\mathbb{P}}_{t,S}$ and $\widehat{\mathbb{P}}_{t,S}$. Here $\widehat{\mathbb{P}}_{t,S} \equiv \widetilde{\mathbb{P}}^{(S)}_{t,S}$ is an equivalent martingale measure with the original asset (e.g., stock) price process $\{S_t\}_{t\ge 0}$ as the numéraire. We have

$$\widehat{C}_{\mathcal{V}}(\tau, m) \equiv \frac{C_{\mathcal{V}}(\tau, S; K, r)}{S} = \widehat{\mathbb{P}}_{t,S}(S_T^{\mathcal{V}} > K) - e^{-m}\widetilde{\mathbb{P}}_{t,S}(S_T^{\mathcal{V}} > K), \tag{5}$$

where t is the current time, T is the expiry time, $\tau = T - t$ is the time to maturity and $m \equiv \ln(S/K) + r\tau$ is the log-forward moneyness.[5] For the randomised G process with $\theta = n \in \mathbb{N}$, we have (call price divided by the spot S):

$$\begin{aligned}\widehat{C}_{G(n,\lambda)}(\tau, m) &= (1 - e^{-m})^+ + \frac{\sqrt{|m|}}{\sqrt{\pi}}\left(\frac{\lambda\tau}{8 + \lambda\tau}\right)^{1/4} e^{-m/2} \\ &\times \sum_{k=0}^{n-1} \frac{1}{k!}\left(\frac{2|m|}{\sqrt{\lambda\tau}\sqrt{8 + \lambda\tau}}\right)^k \mathrm{K}_{k+1/2}\left(\frac{|m|}{2}\frac{\sqrt{8 + \lambda\tau}}{\sqrt{\lambda\tau}}\right),\end{aligned} \tag{6}$$

where $(x)^+ \equiv \max\{x, 0\}$. For the randomised IG process with $\theta = n \in \mathbb{N}$, we have

$$\widehat{C}_{IG(n,\lambda)}(\tau, m) = 1 - \frac{(m^2 + 2\lambda\tau)^{1/4}}{\sqrt{\pi}} e^{-m/2} \sum_{k=0}^{n-1} \frac{1}{k!}\left(\frac{\lambda\tau}{2\sqrt{m^2 + 2\lambda\tau}}\right)^k \mathrm{K}_{k-1/2}\left(\frac{\sqrt{m^2 + 2\lambda\tau}}{2}\right). \tag{7}$$

We derive general formulas for the main Greeks of a European vanilla call option under randomisation. The general formulas are summarised in Table 1.

It can be shown that the option prices in (6) and (7) retain the symmetry property (see Renault & Touzi, 1996, Prop. 3.1),

$$\widehat{C}_{\mathcal{V}}(\tau, m) = (1 - e^{-m}) + e^{-m}\widehat{C}_{\mathcal{V}}(\tau, -m),$$

and exhibit symmetric smiles in the BS implied volatility. In Figure 3, we can see that for given $\lambda > 0$, $\tau > 0$ and log-forward moneyness m, the BS implied volatility is increasing in θ, and deep in- (and out-) of-the-money option (i.e., large values of m in absolute term) prices are more sensitive to the parameter θ than near in- (and out-) of-the-money option (i.e., small vale of m in absolute term) prices. In Figure 4, we can see that for given $\lambda > 0$, $\tau > 0$ and log-forward moneyness m, the BS implied volatility is decreasing in θ, and deep in- (and out-) of-the-money option prices are less sensitive to the parameter θ than near in- (and out-) of-the-money options. Both figures show symmetric smile effects. We can also see that the BS implied volatility under the gamma randomisation exhibits the V-shaped (i.e., locally concave) smile. In contrast, the BS implied volatility under the inverse gamma randomisation displays the U-shaped (i.e., locally convex) smile. We will show in the next section that the inverse gamma randomisation model calibrates well to some U-shaped market volatility. Hence, it may be helpful for practitioners to employ this model. However, the gamma randomisation model does not commonly fit well as we rarely see market volatility with concave smiles in practice.

[4] For $\theta \notin \mathbb{N}$, we can derive the at-the-money forward (ATMF) option prices in closed-form in terms of the hypergeometric functions. The reader may refer to Appendix 7 for the details.

[5] Throughout, we denote $m \equiv m(S, K, \tau) = \ln(S/K) + r\tau$ to avoid clutter.

4 Numerical Example

In this section, we calibrate our models to some market option data. We extracted the market data for the Coca-Cola European call options with spot time on April 2, 2019. The market data contains 354 sample data points with 15 distinct values of the maturity time. The market volatility in the data set exhibits pronounced smiles across different strikes for short times to maturity and skewed smiles for long times to maturity. We decided to compare the performance of the new models with the SABR model because the latter admits a closed-form yet simple celebrated formula for approximate implied volatility. We calibrated the models to the market data among classes consisting of all observations with the same maturity times because the SABR model calibrates well at a single maturity but does not calibrate well at multiple maturities (Wu, 2012). The summary of the market data used here you can found in Table 2. The reader may refer to Tables 3, 4, 5 and Figures 5, 6, 7, 8 for the results.

Suppose that V_i^*, Σ_i^* are the observed market option price and market volatility respectively for $i=1,\ldots,N_\tau$ where $N_\tau = \#\mathcal{S}_\tau$ is the number of observations with maturity time τ, and τ_i, K_i are the corresponding maturity time and strike price. Define $T=\{\tau_i : i=1,\ldots,N\}$ as the collection of maturity times in the data set arranged in increasing order. Let $\mathcal{S}_\tau = \{i \mid \tau_i = \tau \in T\}$ be the collection of observations with maturity time $\tau \in T$. For each τ, we use the usual root mean squared error (RMSE) as a loss function $L(\theta,\lambda)$ for the model calibration under the gamma and the inverse gamma randomisation:

$$L_\tau(\theta,\lambda) = \sqrt{\frac{\sum_{i\in\mathcal{S}_\tau}\left(V(\tau_i,S;K_i)-V_i^*\right)^2}{N_\tau}}; \quad \tau \in T,$$

where $N_\tau = \#\mathcal{S}_\tau$ is the number of observations with maturity time τ, and τ_i, K_i are the corresponding maturity time and strike price. Alternatively, for the SABR model, we use a formula from Hagan et al. (2002), denoted by σ_{SABR}, to find optimal values of parameters that minimise the difference between the corresponding BS implied volatility and the market volatility in the RMSE sense. Hence, the loss function for the SABR model calibration is:

$$L_\tau(\alpha,\beta,\sigma,\rho) = \sqrt{\frac{\sum_{i\in\mathcal{S}_\tau}\left(\sigma_{\mathrm{SABR}}(\tau_i,S,\sigma;K_i)-\Sigma_i^*\right)^2}{N_\tau}}; \quad \tau \in T,$$

For the SABR model parameters, we attempted to find optimal values for the parameters $(\alpha,\sigma,\rho)\equiv(\alpha(\beta),\sigma(\beta),\rho(\beta))$ across different values of $\beta\in[-1,0]$, and find the optimal value of β by comparing the associated RMSEs.[6] We found that $\beta=-1$ gave the lowest RMSE.

Based on Tables 3, 4, and 5, we found that: the inverse gamma randomisation performs better than the gamma randomisation because the RMSE is smaller for fixed τ. Figures 5, 6, 7, and 8 suggest that the inverse gamma randomisation performs quite well for short maturity times, and the SABR model fits almost perfectly.

5 Conclusion

In this paper, we constructed the randomised GBM processes under the gamma and the inverse gamma randomisation, namely the randomised G and IG processes. We observed that both processes had thicker tails than the GBM process, and the randomised IG process had the heaviest tails among the three. We obtained explicit no-arbitrage pricing formulas for European vanilla call options with

[6] In our data set, we saw that β was not a robust parameter since the optimal value for β varies with different initial values of β. So we used the calibration method in Hagan et al. (2002) to find β in advance. There are different approaches for the SABR model calibration, see e.g., West (2005).

integer-valued shape parameter and ATMF option prices with real-valued shape parameter. Surprisingly, the pricing formulas presented in this paper are even simpler than the classical GBM model as they are expressed as elementary analytical functions. The option prices were also obtained numerically in an efficient manner. The European-style option prices under the new processes exhibit symmetric smiles in the log-forward moneyness. We calibrated the randomised GBM models and the SABR model to the actual market option data set from Coca-Cola. We found that the inverse gamma randomisation fitted well, especially for short maturity times.

Further applications of the randomised models will be discussed in other planned future papers. We will provide analytical extensions that take into account imposed killing, leading to closed-form formulas for specific exotic options under the randomised models. We will build a randomisation framework in a multi-asset economy and examine the analytical tractability of other complex derivatives for payoffs, depending on two or more assets.

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Table 1
Greeks of a European vanilla call option under randomisation

Name	**Notation**	**Formula (Note: $m \equiv \ln\frac{S}{K} + r\tau$)**
Delta	$\Delta_\mathcal{V} \equiv \frac{\partial C_\mathcal{V}}{\partial S}$	$\frac{1}{2}\int_{\Omega_\mathcal{V}} \mathrm{erfc}\left(-\frac{m+\frac{1}{2}v\tau}{\sqrt{2v\tau}}\right)\mu_\mathcal{V}(dv)$
Gamma	$\Gamma_\mathcal{V} \equiv \frac{\partial^2 C_\mathcal{V}}{\partial S^2}$	$\frac{1}{S}\int_{\Omega_\mathcal{V}} \frac{1}{\sqrt{2\pi v\tau}} e^{-(m+\frac{1}{2}v\tau)^2/2v\tau}\mu_\mathcal{V}(dv)$
Rho	$\rho_\mathcal{V} \equiv \frac{\partial C_\mathcal{V}}{\partial r}$	$S\tau e^{-m}\frac{1}{2}\int_{\Omega_\mathcal{V}} \mathrm{erfc}\left(-\frac{m-\frac{1}{2}v\tau}{\sqrt{2v\tau}}\right)\mu_\mathcal{V}(dv)$
Theta	$\Theta_\mathcal{V} \equiv -\frac{\partial C_\mathcal{V}}{\partial \tau}$	$\frac{1}{2}S\int_{\Omega_\mathcal{V}} \frac{\sqrt{v}}{\sqrt{2\pi\tau}} e^{-(m+\frac{1}{2}v\tau)^2/2v\tau}\mu_\mathcal{V}(dv) - \frac{r\rho_\mathcal{V}}{\tau}$

Source: The authors.

Table 2
Set of parameters and stopping criterion to be used for calibrating to the market data

Variable	**Description**	**Value**
S	spot price	46.57
r	constant risk-free rate	0%
τ	maturity times (in years)	$0.008 \sim 1.792$
K	strike prices	$23 \sim 65$
TolX	termination tolerance on the current value	10^{-6}
TolFun	termination tolerance on the function value	10^{-6}

Source: The authors.

Table 3
Optimal values of θ and λ under the gamma randomisation (Note that we can only compare the RMSE with the inverse gamma randomisation for fixed τ, but we cannot compare the RMSE across different values of τ because the sample size differs across maturity times)

τ	N_τ	θ	λ	**RMSE**	**Time**
0.008	33	0.095	0.575	0.041	71.194
0.027	33	0.108	0.334	0.059	91.963
0.044	35	0.176	0.171	0.070	8.189
0.066	19	0.405	0.104	0.083	1.932
0.085	15	0.214	0.215	0.159	3.123
0.104	15	0.087	0.720	0.295	40.699
0.123	24	0.193	0.228	0.121	5.902
0.219	32	0.153	0.276	0.162	15.721
0.373	31	0.369	0.089	0.169	4.230
0.468	29	0.322	0.105	0.173	5.593
0.622	24	0.482	0.067	0.175	3.156
0.795	17	2.669	0.009	0.184	1.989
1.216	16	3.245	0.007	0.186	1.837
1.466	14	2.070	0.012	0.227	2.087
1.792	17	11.021	0.002	0.173	2.154

Source: The authors.

Table 4
Optimal values of θ *and* λ *under the inverse gamma randomisation (Note that we can only compare the RMSE with the gamma randomisation for fixed* τ*, but we cannot compare the RMSE across different values of* τ *because the sample size differs across maturity times)*

τ	N_τ	θ	λ	**RMSE**	**Time**
0.008	33	0.719	0.002	0.032	6.087
0.027	33	0.827	0.002	0.051	3.460
0.044	35	0.877	0.003	0.062	3.569
0.066	19	1.227	0.014	0.079	2.332
0.085	15	0.885	0.005	0.147	2.107
0.104	15	0.672	0.002	0.280	2.533
0.123	24	0.923	0.006	0.106	2.788
0.219	32	0.799	0.003	0.135	3.897
0.373	31	0.979	0.006	0.147	2.903
0.468	29	0.926	0.005	0.141	3.787
0.622	24	1.091	0.009	0.153	3.789
0.795	17	2.406	0.035	0.181	2.338
1.216	16	2.962	0.048	0.183	2.697
1.466	14	1.861	0.024	0.217	2.310
1.792	17	8.016	0.155	0.173	3.485

Source: The authors.

Table 5
Optimal values of α, σ *and* ρ *under the SABR model (Note that we do not display the RMSEs here because the units associated from the SABR model is different from the randomised GBM models)*

τ	N_τ	α	σ	ρ	**Time**
0.008	33	21.729	2.994	–0.502	0.435
0.027	33	10.616	3.643	–0.560	0.574
0.044	35	7.711	4.204	–0.619	0.353
0.066	19	4.691	6.793	–0.465	0.329
0.085	15	5.040	6.239	–0.610	0.341
0.104	15	5.967	5.473	–0.704	0.344
0.123	24	3.633	6.119	–0.570	0.337
0.219	32	2.916	5.424	–0.604	0.360
0.373	31	1.895	5.701	–0.535	0.159
0.468	29	1.631	5.680	–0.385	0.392
0.622	24	1.147	6.256	–0.341	0.378
0.795	17	1.001	6.032	–0.425	0.359
1.216	16	0.673	6.272	–0.242	0.371
1.466	14	0.782	5.940	–0.154	0.303
1.792	17	0.467	6.371	–0.058	0.331

Source: The authors.

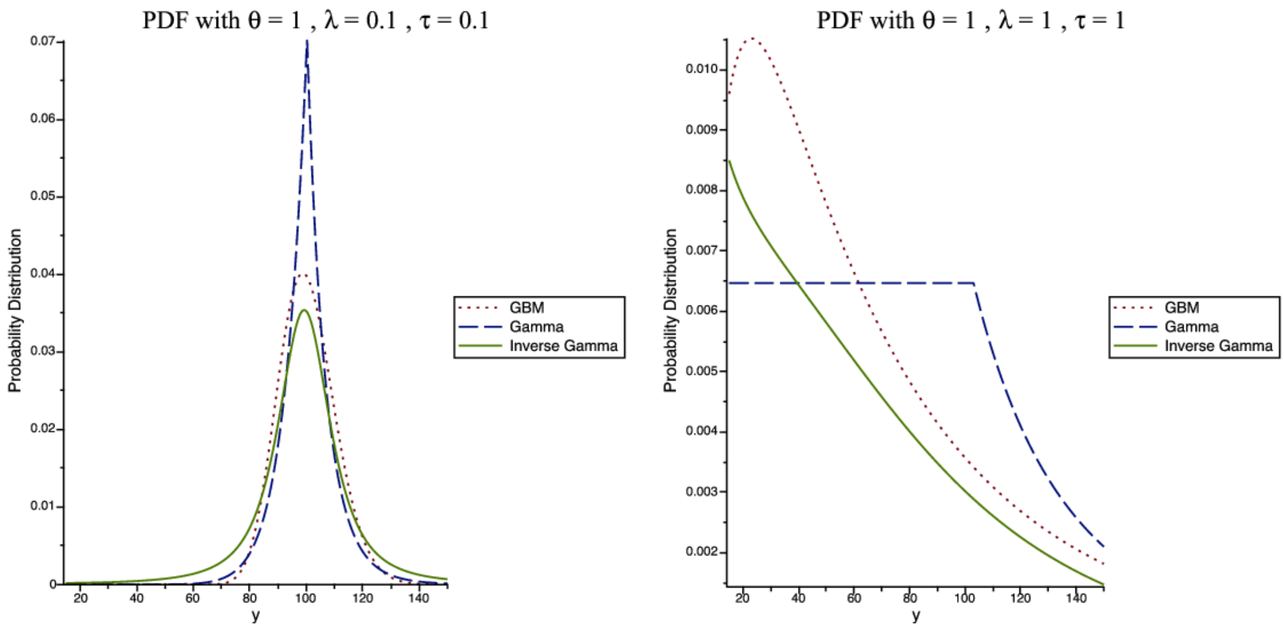

Figure 1. Plots of the transition PDFs for the process S_t, $S_t^{G(\theta,\lambda)}$, and $S_t^{IG(\theta,\lambda)}$, where $S=100$, $r=0.03$, and $v=0.1$

Source: The authors.

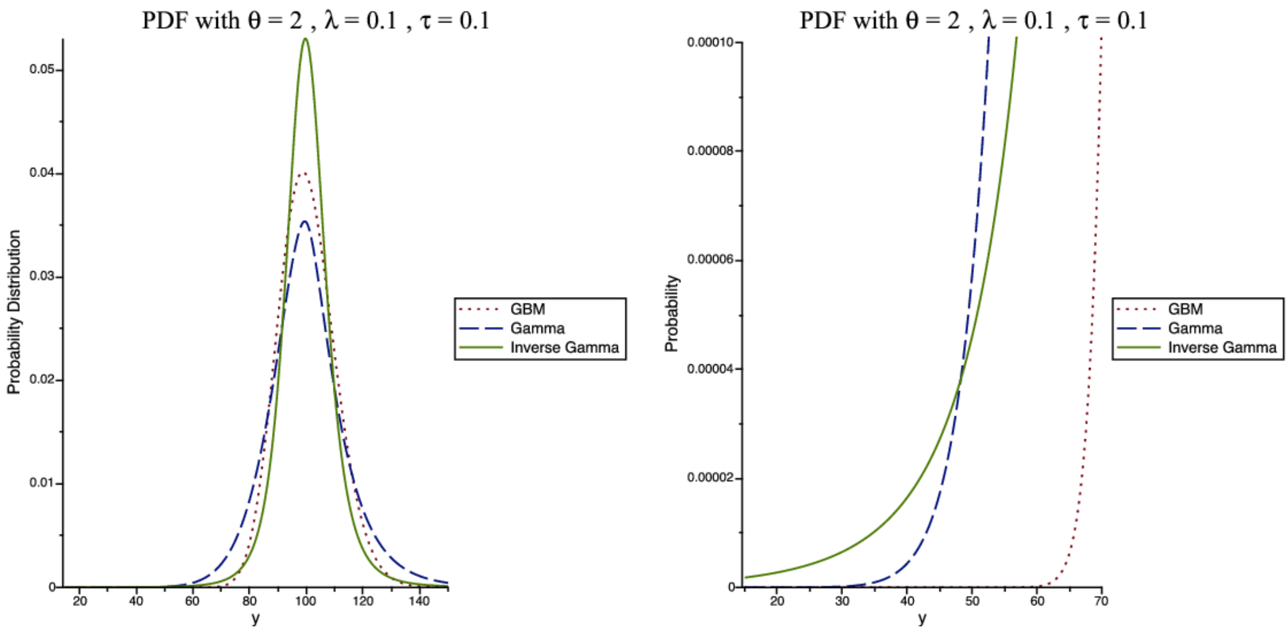

Figure 2. Plots of the transition PDFs for the process S_t, $S_t^{G(\theta,\lambda)}$, and $S_t^{IG(\theta,\lambda)}$, where $S=100$, $r=0.03$, and $v=0.1$

Source: The authors.

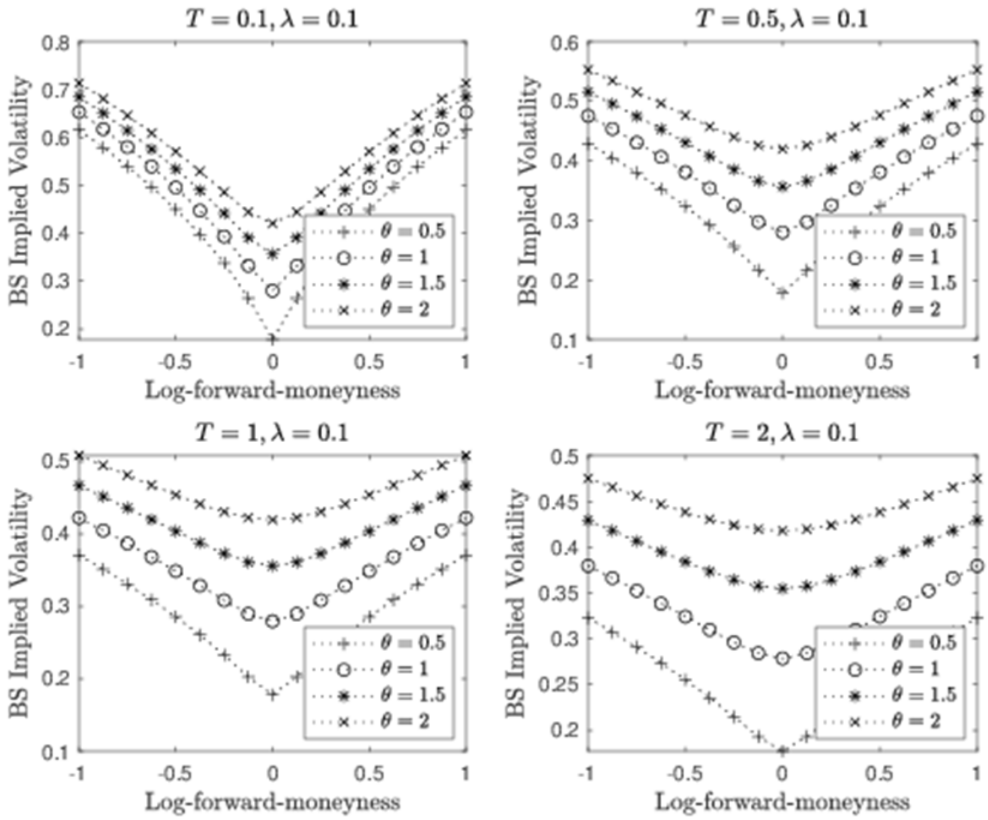

Figure 3. BS implied volatility of a European vanilla call option under the gamma randomisation

Source: The authors.

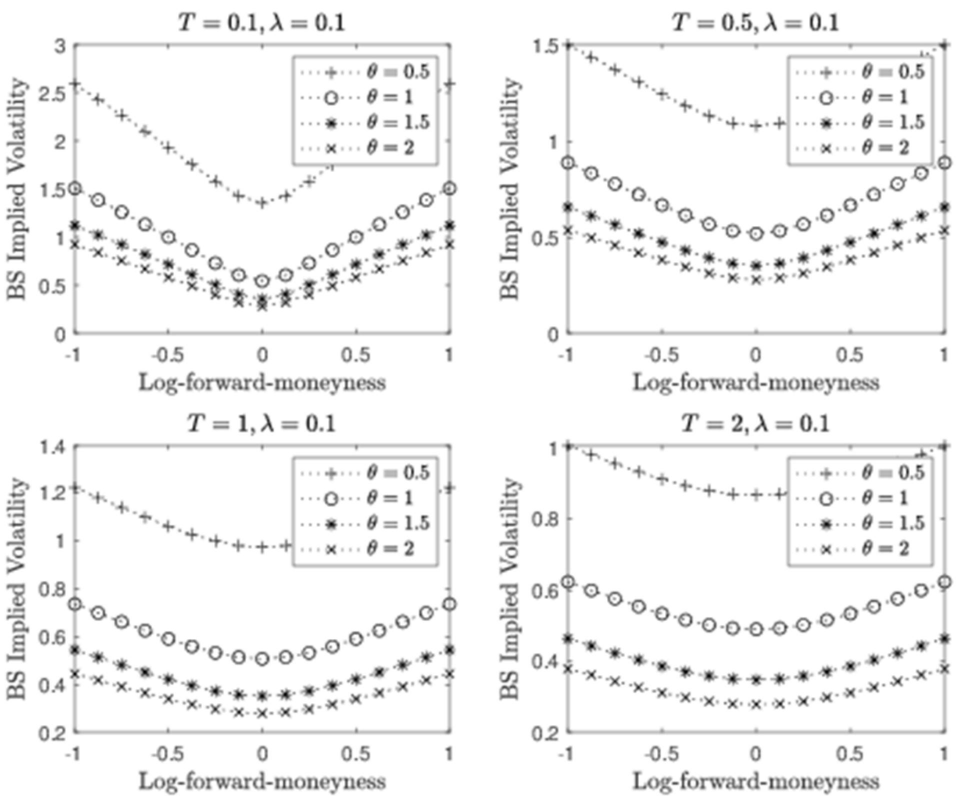

Figure 4. BS implied volatility of a European vanilla call option under the inverse gamma randomisation

Source: The authors.

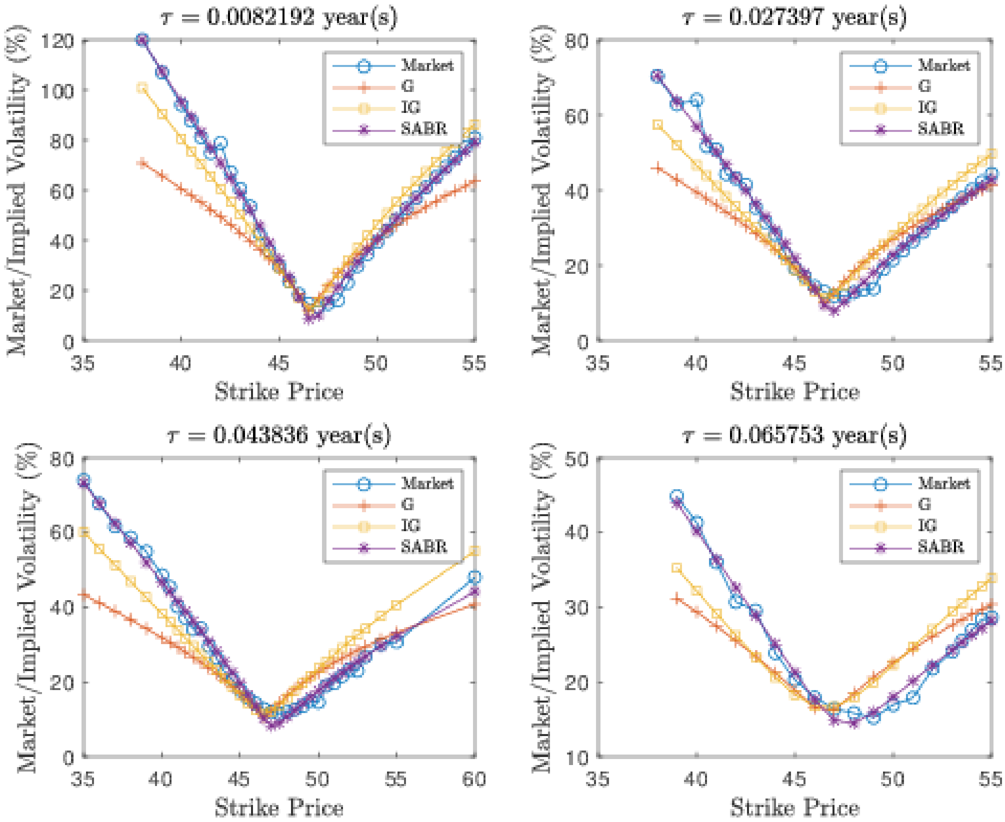

Figure 5. 2D Implied volatility plots for $\tau = 0.008 \sim 0.066$ years

Source: The authors.

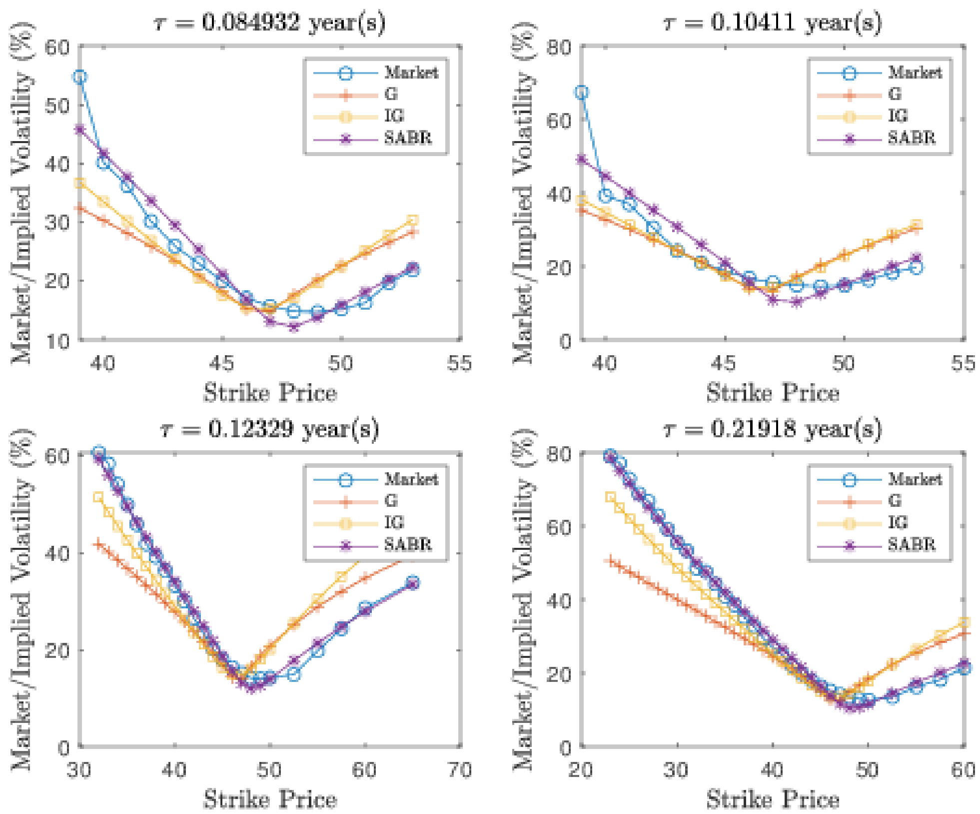

Figure 6. 2D Implied volatility plots for $\tau = 0.085 \sim 0.219$ years

Source: The authors.

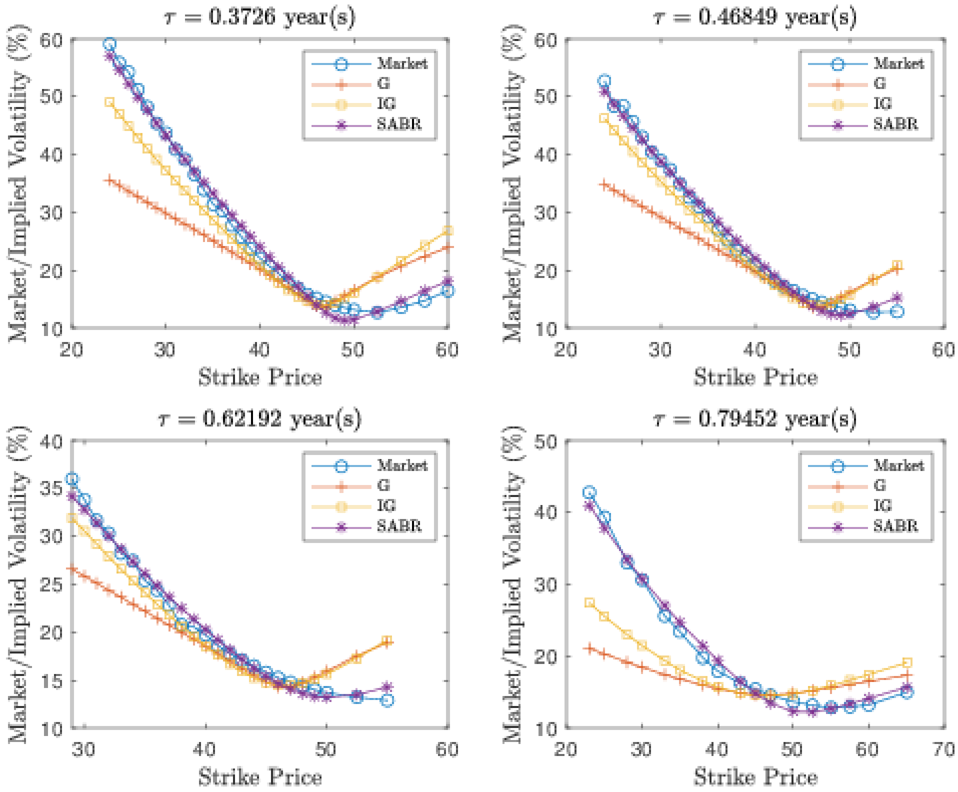

Figure 7. 2D Implied volatility plots for $\tau = 0.373 \sim 0.795$ years

Source: The authors.

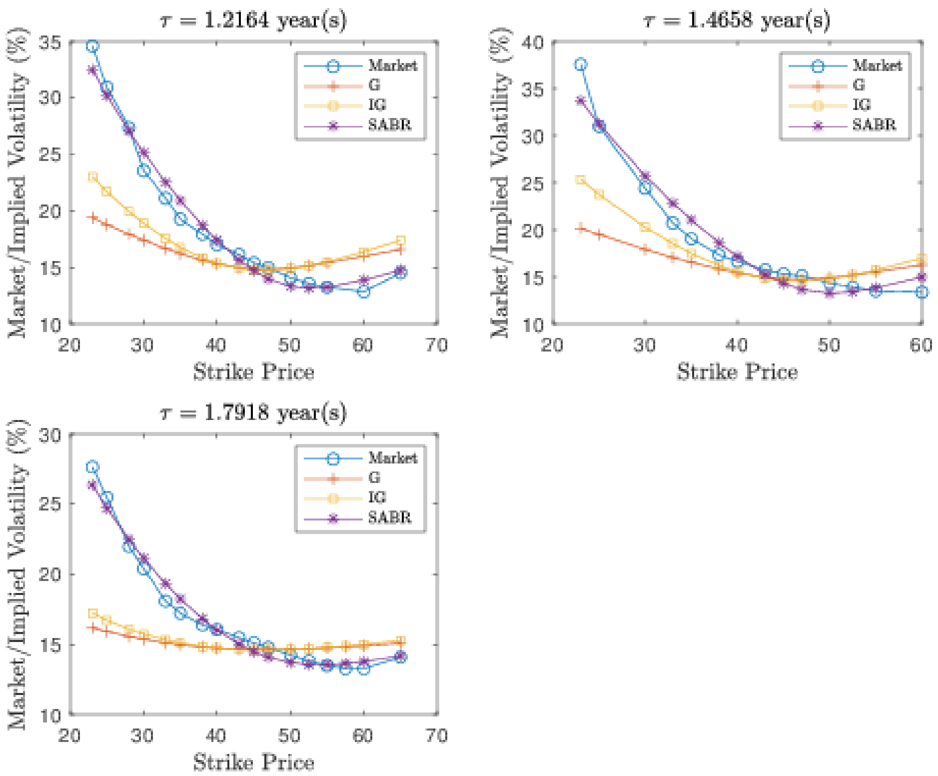

Figure 8. 2D Implied volatility plots for $\tau = 1.216 \sim 1.792$ years

Source: The authors.

Appendix

A.1 Proof of the Exact Pricing Formula with Integer-valued Shape Parameter

Let us take $\Lambda(S_T)=1_{\{S_t>K\}}$ with $K>0$, where $1_{\mathcal{A}}$ is the indicator function of some event $\mathcal{A}$. By (2) we have the following risk-neutral conditional probability that the asset price is above the strike K at the time T:

$$\widetilde{\mathbb{P}}_{t,S}(S_T^{\mathcal{V}}>K)=\int_{\Omega_{\mathcal{V}}}\mathcal{N}\left(\frac{m-\frac{1}{2}v\tau}{\sqrt{v\tau}}\right)\mu_{\mathcal{V}}(dv)=\frac{1}{2}\int_{\Omega_{\mathcal{V}}}\operatorname{erfc}\left(-\frac{m-\frac{1}{2}v\tau}{\sqrt{2v\tau}}\right)\mu_{\mathcal{V}}(dv),$$

where erfc is the complementary error function. We state another useful integral formula (see Prudnikov, Brychkov, & Marichev, 1986, Eq. 2.8.9.7):[1]

$$\int_0^\infty x^n e^{-px}\operatorname{erfc}\left(c\sqrt{x}+\frac{b}{\sqrt{x}}\right)dx=\frac{2(n)!}{p^{n+1}}\{1_{\{b<0\}}+\frac{\sqrt{|b|}(c^2+p)^{1/4}}{\sqrt{\pi}}e^{-2bc}\sum_{k=0}^{n}\frac{p^k}{k!}\left(\frac{b^2}{c^2+p}\right)^{k/2}$$
$$\times[\operatorname{sgn}(b)\mathrm{K}_{k-1/2}(2|b|\sqrt{c^2+p})-\frac{c}{\sqrt{c^2+p}}\mathrm{K}_{k+1/2}(2|b|\sqrt{c^2+p})]\},\tag{8}$$

where sgn is the sign function with $\operatorname{sgn}(0)=1$. We can use (8) to obtain analytical formulas for the randomised processes in the case with integer-valued $\theta=n\in\mathbb{N}$. For the randomised G process, we have:

$$\widetilde{\mathbb{P}}_{t,S}(S_T^{G(n,\lambda)}>K)=1_{\{m>0\}}-\frac{\sqrt{|m|}}{2\sqrt{\pi}}\left(\frac{8+\lambda\tau}{\lambda\tau}\right)^{1/4}e^{m/2}\sum_{k=0}^{n-1}\frac{1}{k!}\left(\frac{2|m|}{\sqrt{\lambda\tau}\sqrt{8+\lambda\tau}}\right)^k$$
$$\times\left[\operatorname{sgn}(m)\mathrm{K}_{k-1/2}\left(\frac{|m|}{2}\frac{\sqrt{8+\lambda\tau}}{\sqrt{\lambda\tau}}\right)+\frac{\sqrt{\lambda\tau}}{\sqrt{8+\lambda\tau}}\mathrm{K}_{k+1/2}\left(\frac{|m|}{2}\frac{\sqrt{8+\lambda\tau}}{\sqrt{\lambda\tau}}\right)\right].\tag{9}$$

For the randomised IG process, upon changing the integration variable, we have:

$$\widetilde{\mathbb{P}}_{t,S}(S_T^{IG(n,\lambda)}>K)=\frac{(m^2+2\lambda\tau)^{1/4}}{2\sqrt{\pi}}e^{m/2}\sum_{k=0}^{n-1}\frac{1}{k!}\left(\frac{\lambda\tau}{2\sqrt{m^2+2\lambda\tau}}\right)^k$$
$$\times\left[\mathrm{K}_{k-1/2}\left(\frac{\sqrt{m^2+2\lambda\tau}}{2}\right)+\frac{m}{\sqrt{m^2+2\lambda\tau}}\mathrm{K}_{k+1/2}\left(\frac{\sqrt{m^2+2\lambda\tau}}{2}\right)\right].\tag{10}$$

Now, we consider the risk-neutral conditional probability $\widehat{\mathbb{P}}\equiv\widetilde{\mathbb{P}}^{(S)}$ under an equivalent martingale measure with the asset price process $\{S_t\}_{t\geq 0}$ as the numéraire, where

$$\widehat{\mathbb{P}}_{t,S}(S_T^{\mathcal{V}}>K)\equiv\widetilde{\mathbb{E}}_{t,S}\left[\frac{S_T^{\mathcal{V}}1_{\{S_T^{\mathcal{V}}>K\}}/B_T}{S_t^{\mathcal{V}}/B_t}\right]=\frac{1}{2}\int_{\Omega_{\mathcal{V}}}\operatorname{erfc}\left(-\frac{m+\frac{1}{2}v\tau}{\sqrt{2v\tau}}\right)\mu_{\mathcal{V}}(dv).\tag{11}$$

[1] The integral formula is valid for $\Re(p)>0, |\arg(c)|<\frac{\pi}{4}$. Moreover, it would be valid for $\Re(c^2+p)>0$ if $\Re(c)>0$.

where $B_t = e^{rt}$ is the bank account value at the time t. For the randomised G process $\{S_t^{G(n,\lambda)}\}_{t\geq 0}, n\in\mathbb{N}$, by using (11) and (8), we have:

$$\widehat{\mathbb{P}}_{t,S}(S_T^{G(n,\lambda)} > K) = 1_{\{m>0\}} - \frac{\sqrt{|m|}}{2\sqrt{\pi}}\left(\frac{8+\lambda\tau}{\lambda\tau}\right)^{1/4} e^{-m/2}\sum_{k=0}^{n-1}\frac{1}{k!}\left(\frac{2|m|}{\sqrt{\lambda\tau}\sqrt{8+\lambda\tau}}\right)^k \times\left[\operatorname{sgn}(m)\mathrm{K}_{k-1/2}\left(\frac{|m|}{2}\frac{\sqrt{8+\lambda\tau}}{\sqrt{\lambda\tau}}\right) - \frac{\sqrt{\lambda\tau}}{\sqrt{8+\lambda\tau}}\mathrm{K}_{k+1/2}\left(\frac{|m|}{2}\frac{\sqrt{8+\lambda\tau}}{\sqrt{\lambda\tau}}\right)\right]. \tag{12}$$

By substituting (9) and (12) into (5), we obtain (6). For the randomised IG process $\{S_t^{IG(n,\lambda)}\}_{t\geq 0}$, $n\in\mathbb{N}$, by using (11) and (8) we have:

$$\widehat{\mathbb{P}}_{t,S}(S_T^{IG(n,\lambda)} > K) = 1 - \frac{(m^2+2\lambda\tau)^{1/4}}{2\sqrt{\pi}} e^{-m/2}\sum_{k=0}^{n-1}\frac{1}{k!}\left(\frac{\lambda\tau}{2\sqrt{m^2+2\lambda\tau}}\right)^k \times\left[\mathrm{K}_{k-1/2}\left(\frac{\sqrt{m^2+2\lambda\tau}}{2}\right) - \frac{m}{\sqrt{m^2+2\lambda\tau}}\mathrm{K}_{k+1/2}\left(\frac{\sqrt{m^2+2\lambda\tau}}{2}\right)\right]. \tag{13}$$

By substituting (10) and (13) into (5), we obtain (7).

A.2 The Exact Pricing Formulas for ATMF Options

The price of an ATMF (i.e., $m \equiv \ln(S/K) + r\tau = 0$) European vanilla call option under the GBM model, with variance randomised according to the probability measure $\mu_{\mathcal{V}}$, can be expressed as:

$$\widehat{C}_{\mathcal{V}}(\tau,0) = \int_{\Omega_{\mathcal{V}}} \operatorname{erf}\left(\frac{\sqrt{v\tau}}{2\sqrt{2}}\right)\mu_{\mathcal{V}}(dv).$$

Where erf is the error function. We use the above equation to derive the pricing formulas for ATMF options explicitly under the gamma and inverse gamma randomisation for shape parameter $\theta\in\mathbb{R}_+$.

Proposition 1 *The price (divided by spot S) of an ATMF European vanilla call option under the gamma randomisation is:*

$$\widehat{C}_{G(\theta,\lambda)}(\tau,0) = 1 - \frac{\Gamma(\theta+\frac{1}{2})}{\sqrt{\pi}\Gamma(\theta+1)}\left(\frac{8}{\lambda\tau}\right)^{\theta} {}_2\mathrm{F}_1(\theta,\theta+\frac{1}{2};\theta+1,-\frac{8}{\lambda\tau}),$$

where ${}_p\mathrm{F}_q(\mathbf{a};\mathbf{b};z)$ is the generalised hypergeometric function.

Proof

We first make a note that the incomplete gamma function can be expressed in terms of the Kummer function of the first kind.i.e.,

$$\gamma(\theta,x) = \theta^{-1}x^{\theta}\ {}_1\mathrm{F}_1(\theta;\theta+1;-x).$$

Hence, we have

$$\widehat{C}_{G(\theta,\lambda)}(\tau,0)=1-\frac{1}{\sqrt{\pi}\Gamma(\theta+1)}\left(\frac{8}{\lambda\tau}\right)^{\theta}\int_0^{\infty}\ {}_1F_1\left(\theta;\theta+1;-\frac{8y}{\lambda\tau}\right)y^{\theta-1/2}e^{-y}dy.$$

And an integral representation of a generalised hyperbolic function is:[2]

$${}_{p+1}\mathrm{F}_q\left(\begin{matrix}a_0,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)=\frac{1}{\Gamma(a_0)}\int_0^{\infty}s^{a_0-1}e^{-s}\ \ {}_p\mathrm{F}_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};zs\right)ds.$$

From the integral representation above, we obtain the final expression.

Proposition 2 *The price (divided by spot S) of an ATMF European vanilla call option under the inverse gamma randomisation is:*

$$\widehat{C}_{IG(\theta,\lambda)}(\tau,0)=\sqrt{\frac{\lambda\tau}{2\pi}}\frac{\Gamma\left(\theta-\frac{1}{2}\right)}{\Gamma(\theta)}\ \ {}_1\mathrm{F}_2\left(\frac{1}{2};\frac{3}{2},\frac{3}{2}-\theta;\frac{\lambda\tau}{8}\right)$$
$$+\left(\frac{\lambda\tau}{8}\right)^{\theta}\frac{\Gamma\left(\frac{1}{2}-\theta\right)}{\sqrt{\pi}\Gamma(\theta+1)}\ \ {}_1\mathrm{F}_2\left(\theta;\theta+1,\theta+\frac{1}{2};\frac{\lambda\tau}{8}\right).$$

Proof

We first make a note of an integral representation of the Kummer function of the first kind

$${}_1\mathrm{F}_1(a,b,c)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{cu}u^{a-1}(1-u)^{b-a-1}du.$$

Hence, we have

$$\widehat{C}_{IG(\theta,\lambda)}(\tau,0)=\frac{2}{\sqrt{\pi}\Gamma(\theta)}\left(\frac{\lambda\tau}{8}\right)^{1/4+\theta/2}\int_0^1 u^{\theta/2-3/4}\mathrm{K}_{\theta-1/2}\left(\sqrt{\frac{\lambda\tau u}{2}}\right)du.$$

Now use the fact that modified Bessel functions of the second kind can be expressed in terms of generalised hypergeometric functions.i.e.,

$$\mathrm{K}_{\theta}(x)=\frac{\Gamma(\theta)}{2}\left(\frac{x}{2}\right)^{-\theta}\ {}_0\mathrm{F}_1(;-\theta+1;\frac{x^2}{4})+\frac{\Gamma(-\theta)}{2}\left(\frac{x}{2}\right)^{\theta}\ {}_0\mathrm{F}_1(;\theta+1;\frac{x^2}{4}).$$

Another integral representation of a generalised hyperbolic function is:[3]

$${}_{p+1}\mathrm{F}_{q+1}\left(\begin{matrix}a_0,\ldots,a_p\\ b_0,\ldots,b_q\end{matrix};z\right)=\frac{\Gamma(b_0)}{\Gamma(a_0)\Gamma(b_0-a_0)}\int_0^1 s^{a_0-1}(1-s)^{b_0-a_0-1}\ \ {}_p\mathrm{F}_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};zs\right)ds.$$

From the integral representation above, we obtain the final expression.

---

[2] The integral formula is valid for $\Re(z)<1,\Re(a_0)>0$.

[3] The integral formula is valid for $\Re(b_0)>\Re(a_0)>0$.

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Заявленный вклад авторов
Джузеппе (Джо) Камполиети — концепция и основные тезисы статьи.
Хиромичи Като — ссылки, анализ данных.
Роман Макаров — вычисления, корректура формул.

The article was submitted on 17.05.2021, reviewed on 16.07.2021, and accepted for publication on 17.08.2021.
The authors read and approved the final version of the manuscript.
Статья поступила в редакцию 17.05.2021; после рецензирования 16.07.2021; принята к публикации 17.08.2021.
Авторы прочитали и одобрили окончательный вариант рукописи.

ORIGINAL PAPER

DOI: 10.26794/2308-944X-2021-9-3-27-51
JEL C6, C61, C63, D81, G1, G2, G12, G13, G22

On Approximate Pricing of Spread Options via Conditional Value-at-Risk

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ABSTRACT
It is widely accepted to use conditional value-at-risk for risk management needs and option pricing. As a rule, there are difficulties in exact calculations of conditional value-at-risk. In the paper, we use the conditional value-at-risk methodology to price spread options, extending some approximation approaches for these needs. Our results we illustrate by numerical calculations which demonstrate their effectiveness. We also show how conditional value-at-risk pricing can help with regulatory needs inspired by the Basel Accords.
***Keywords*:** spread options; conditional value-at-risk; approximation methods; capital constraints; Margrabe market

For citation: **Maksimov, C., & Melnikov, A. (2021). On approximate pricing of spread options via conditional value-at-risk. *Review of Business and Economics Studies, 9*(3),27-51. DOI: 10.26794/2308-944X-2021-9-3-27-51**

ОРИГИНАЛЬНАЯ СТАТЬЯ

Приближенное оценивание опционов обмена акций через ожидаемые потери

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АННОТАЦИЯ
Как правило, существуют сложности для точного определения уровня ожидаемых потерь в целях управления риском и для оценивания опционов. В данной работе использована методология ожидаемых потерь для оценки опционов обмена акций с помощью некоторых приближений. Эффективность результатов наглядно продемонстрирована численными расчетами. Показано, как анализ ожидаемых потерь может помочь в выполнении требований постановлений Базельских соглашений III.
Ключевые слова: варианты распространения; условная стоимость, подверженная риску; методы аппроксимации; нехватка капитала; Рынок Марграбе

1 Introduction

In complete markets, every contingent claim is replicable in the class of self-financing strategies, and its price is unique. However, there is a whole range of arbitrage-free prices in incomplete markets or in markets with constraints. The minimum price that guarantees no underhedging at maturity is equal to the supremum of this price range. The resulting strategy is known as superhedging (see, for instance, El Karoui and Quenez (1995)). An investor can choose to stay within the boundaries of perfect hedging and completely eliminate potential risks by engaging in superhedging strategies. But the cost of such a strategy can be too high to be implemented successfully. A viable alternative is to accept the possibility of a shortfall — the difference between the payoff of the contingent claim and the replicating portfolio at maturity. This approach is usually exploited when there are market constraints on the amount of capital that can be used for hedging. It has practical benefits as regulators frequently require financial institutions to use a certain amount of funds conservatively to be able to meet their obligations. Still, the extra funds saved on hedging can be used more aggressively in an attempt to earn an extra return. Two main approaches have been considered in the literature. The first one includes maximising the probability of a successful hedge. One of the earliest works is by Kulldorf (1993). The author considered a stochastic control problem with a single risky asset whereby an agent aims to reach a particular value of fortune on a finite time interval before first going broke. Browne (1999) expanded upon the results obtained by Kulldorf (1993). The author considered a market setting with several risky securities and determined the optimal policy that maximises the probability of reaching a certain level of wealth before some fixed terminal time. Working in this direction, Foellmer and Leukert (1999) transformed the initial problem into the problem when an optimal strategy maximises the probability of successful hedging. The resulting strategy can be viewed as a dynamic version of the Value-at-Risk (VaR) concept, a popular measure of market risk exposure. The major drawback of the approach is that the size of the potential shortfall is not taken into account. Developing the approach, Foellmer and Leukert (2000) propose to minimise the amount of expected shortfall where some loss function l measures an investor's attitude to the size of the shortfall. The key idea is to use the Neyman-Pearson lemma to modify the original contingent claim so that the modified contingent claim can be perfectly hedged. The authors show that the modified claim's perfect hedging strategy is also the optimal strategy for the initial minimisation problem.

The methodology proposed by Foellmer and Leukert (2000) leaves some space for a choice of the loss function to model the attitude of the investor towards the potential shortfall. Value-at-risk (VaR), being the most popular tool for measuring market risk exposure by practitioners, is a natural choice. However, the use of VaR was severely criticised for failing to predict the scope of the losses during the global financial crisis. The most recent Basel III framework has signified the major shift from VaR to conditional Value-at-Risk (CVaR) as the encouraging measure of risk under stress. According to the Basel Committee on Banking Supervision (2016), the use of CVaR "will help to ensure a more prudent capture of 'tail risk' and capital adequacy during periods of significant financial market stress." CVaR has some beneficial mathematical properties that VaR lacks. First of all, CVaR satisfies the four properties of translation invariance, subadditivity, positive homogeneity, and monotonicity, making it a coherent measure of risk (Artzner et al., 1999)). In general, VaR does not satisfy the subadditivity property unless the joint distribution function of portfolio losses is from a family of elliptical distributions. Another advantage of CVaR over VaR is that it is a spectral measure of risk (Acerbi, 2002)), meaning that it directly relates to the notion of risk-aversion, an essential concept in studying optimal consumption problems through the use of utility functions. One major drawback of CVaR is that it, in its original form, is a hard risk measure to optimise with respect to. According to Brutti Righi and Ceretta (2016), "despite the advantages of ES, this measure is less frequently utilised than VaR because forecasting ES is challenging due to its complex definition", where ES stands for the same concept as CVaR. However, Rockafellar and Uryasev (2000) showed that an intrinsic relation between the two risk measures exists and developed a methodology for optimising an investment portfolio with respect to both VaR and CVaR simultaneously. The central idea is to introduce an auxiliary function F through which VaR and CVaR can be expressed. The properties of convexity and continuous differentiability make the function F "well-behaved" for optimisation tasks. Melnikov and Smirnov (2012) applied the ideas of Foellmer and Leukert (2000) to the case where CVaR represents the loss function l that models the attitude of an agent to risk and considered the following dual problem: minimisation of CVaR when the initial capital is bounded from above, and minimisation of hedging costs subject to a constraint of the amount of CVaR. The authors further used the representation of CVaR as in Rockafellar and Uryasev (2000). The explicit results were obtained within the framework of the Black-Scholes market with a single risky asset.

This paper aims to take a step in the direction of generalising the results obtained by Melnikov and Smirnov (2012) and consider the problem of CVaR-based option pricing within the context of the Margrabe market model with two risky assets. The option type of interest is a plain vanilla spread option. Spread options are broadly used and appear in a wide range of financial markets: as crack spread option in energy markets, as credit spread options in fixed income markets, and as options to exchange one asset for another in equity markets (see Margrabe, 1978; Fischer, 1978). The problem is further complicated in several aspects. For example, a non-trivial aspect of pricing such options requires knowing the probability distribution of the difference between log-normal random variables that do not admit a satisfactory theoretical expression. Hence, some approximation methods are necessary. In particular, the paper utilises the approximate spread option pricing methodology proposed by Bjerksund and Stensland (2006) and an approximation based on the assumption that the difference between two log-normal random variables is normally distributed. Furthermore, CVaR is chosen as the measure of risk to make the paper's results easily applicable by practitioners in the industry.

2 Preliminaries and Existing Approximating Methods

Let $(\Omega, \mathcal{F}, F = (\mathcal{F}(t))_{t\geq 0}, P)$ be a standard stochastic basis with filtration $\mathcal{F}(t)$ that satisfies the usual conditions, and $\mathcal{F}(0) = \{\Omega, \varnothing\}$. Assume that T is the terminal time for all the contingent claims traded on this market. Then the dynamics of the two stock price processes $S_1 = (S_1(t) : t \in [0,T])$ and $S_2 = (S_2(t) : t \in [0,T])$ are assumed to satisfy the following stochastic differential equations (SDEs):

$$\begin{aligned} dS_1(t) &= S_1(t)\left[\mu_1 dt + \sigma_1 dW_1(t)\right], \\ dS_2(t) &= S_2(t)\left[\mu_2 dt + \sigma_2 dW_2(t)\right], \end{aligned} \tag{1}$$

where $W_1 = (W_1(t) : t \in [0,T])$ and $W_2 = (W_2(t) : t \in [0,T])$ are standard Brownian motion processes with correlation coefficient ρ.

The original Margrabe market model only assumed the existence of two risky assets and no bank account. Therefore, we take all the stocks traded in this market as already discounted.

We further assume that the market is arbitrage-free and complete and introduce a unique equivalent martingale measure Q via the Radon-Nikodym derivative:

$$Z(T) = \frac{dQ}{dP}.$$

We say that measure Q is equivalent to measure P if the two measures agree on the sets of measure 0, i.e., if

$$P(w) > 0 \Leftrightarrow Q(w) > 0.$$

The process $Z = (Z(t) : t \in [0,T])$ takes the following functional form (see, for instance, Melnikov (2011)):

$$Z(t) = \exp\left[\phi_1 W_1(t) + \phi_2 W_2(t) - \frac{\sigma_\phi^2}{2} t\right], \tag{2}$$

where

$$\begin{aligned}\phi_1 &= \frac{\sigma_1\mu_2\rho - \sigma_2\mu_1}{\sigma_1\sigma_2\left(1-\rho^2\right)},\\ \phi_2 &= \frac{\sigma_2\mu_1\rho - \sigma_1\mu_2}{\sigma_1\sigma_2\left(1-\rho^2\right)},\\ \sigma_\phi^2 &= \phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2.\end{aligned}$$

Under the risk-neutral probability measure Q, the dynamics of the two assets satisfy

$$\begin{aligned}dS_1(t) &= S_1(t)\sigma_1 dW_1^Q(t),\\ dS_2(t) &= S_2(t)\sigma_2 dW_2^Q(t),\end{aligned} \tag{3}$$

where $W_1^Q = \left(W_1^Q(t): t\in[0,T]\right)$ and $W_2^Q = \left(W_2^Q(t): t\in[0,T]\right)$ are, according to the Girsanov theorem (Shreve (2011)), standard Brownian motion processes with correlation coefficient ρ. We can rewrite the process Z under the measure Q as follows:

$$Z(t) = \exp\left[\phi_1 W_1^Q(t) + \phi_2 W_2^Q(t) - \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)t\right], \tag{4}$$

where

$$\begin{aligned}\theta_1 &= \frac{\mu_1}{\sigma_1},\\ \theta_2 &= \frac{\mu_2}{\sigma_2}.\end{aligned}$$

The general payoff function of a spread option is of the following form:

$$[S_1(T) - S_2(T) - K]^+, \tag{5}$$

where K is a deterministic strike price. The exact pricing formula for the special case when $K=0$ was determined independently by Margrabe (1978) and Fischer (1978). The price of such a contingent claim is given by

$$p = S_1(0)\Phi(d_1) - S_2(0)\Phi(d_2), \tag{5}$$

where

$$\begin{aligned}d_1 &= \frac{\ln\left(\frac{S_1(0)}{S_2(0)}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}},\\ d_2 &= d_1 - \sigma\sqrt{T},\\ \sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho},\end{aligned}$$

and $\Phi(x)$ is the standard normal cumulative distribution function (CDF). To avoid ambiguity, we will refer to the special case of equation (5) as an option to exchange one asset for another one; and as a spread option otherwise.

However, it is generally accepted that the probability density function (PDF) of linear combinations of log-normal random variables does not have a closed form. Approximations of the distribution of sums of log-normal random variables exist in the literature; see, for example, Mehta et al. (2007), Cobb and Rumi (2012), Hcine and Bouallegue (2015). Less is known about the distribution of the difference between correlated log-normal random variables. Lo (2012) proposed the Lie-Trotter operator splitting method and found that a shifted log-normal process governs the difference between two log-normal random variables. A more recent work by Gulisashvili and Tankov (2016) considers the tail behaviour of the distributions of linear combinations of log-normal random variables explicitly. The results of the paper allow approximating the probabilities of tail events directly. The authors further provide insights into how these findings can be applied in the domain of risk management.

Thus, only approximate pricing formulas for equation (5) exist. Carmona and Durrleman (2003) provide a thorough overview of spread option pricing methodologies. However, while most of the approximations to equation (5) that exist in the literature provide accurate estimates, these are not always easily transferable to the domain of risk management due to their complexity. For the purposes of this paper, we will work around the idea of approximating the difference between two log-normal random variables using a normal distribution. According to Carmona and Durrleman (2003), "computing histograms of historical spread values shows that the marginal distribution of a spread at a given time extends on both tails, and surprisingly enough, that the normal distribution can give a reasonable fit." It allows us to price an option with a payoff as in equation (5) in the approximate form, similar to equation (6). Consider the difference between the two stock prices at maturity:

$$S_1(T)-S_2(T)=S_1(0)\exp\left[-\frac{\sigma_1^2 T}{2}+\sigma_1 W_1^Q(T)\right]-S_2(0)\exp\left[-\frac{\sigma_2^2 T}{2}+\sigma_2 W_2^Q(T)\right]. \quad (7)$$

The above expression represents the difference between two log-normal random variables distribution of which is not log-normal and generally has not been determined. By applying Taylor series expansion to the exponents,

$$S_1(T)-S_2(T)=S_1(0)-S_2(0)+S_1(0)z_1-S_2(0)z_2+S_1(0)\sum_{n=2}^{\infty}\frac{z_1^n}{n!}-S_2(0)\sum_{n=2}^{\infty}\frac{z_2^n}{n!}, \quad (8)$$

where

$$\begin{aligned} z_1 &= -\frac{\sigma_1^2 T}{2}+\sigma_1 W_1^Q(T)\sim N\left(-\frac{\sigma_1^2 T}{2},\sigma_1^2 T\right),\\ z_2 &= -\frac{\sigma_2^2 T}{2}+\sigma_2 W_2^Q(T)\sim N\left(-\frac{\sigma_2^2 T}{2},\sigma_2^2 T\right). \end{aligned}$$

Equation (8) represents a normal random variable plus an error term in the amount of $S_1(0)\sum_{n=2}^{\infty}\frac{z_1^n}{n!}-S_2(0)\sum_{n=2}^{\infty}\frac{z_2^n}{n!}$. The price of the option with a payoff as in equation (5) can then be approximated as follows:

$$p=S_1(0)\Phi(d_1)-S_2(0)\Phi(d_2)-K\Phi(d_3), \quad (9)$$

where

$$\begin{aligned} d_1 &= \frac{-K-m}{\sigma}+\sigma_1\rho_1\sqrt{T},\\ d_2 &= \frac{-K-m}{\sigma}+\sigma_2\rho_2\sqrt{T},\\ d_3 &= \frac{-K-m}{\sigma},\\ \rho_1 &= \frac{\left[S_1(0)\sigma_1-S_2(0)\sigma_2\rho\right]\sqrt{T}}{\sigma},\\ \rho_2 &= \frac{\left[S_1(0)\sigma_1\rho-S_2(0)\sigma_2\right]\sqrt{T}}{\sigma}, \end{aligned}$$

and where $S_1(T)-S_2(T)\approx\gamma\sim N(m,\sigma^2)$, i.e., where the difference $S_1(T)-S_2(T)$ in the indicator function of the option, exercise event is replaced by a normal random variable γ with mean m and variance σ^2. We can use the moment matching technique to calculate the moments of γ. Consider the mean m,

$$
\begin{aligned}
m &= E_Q\left[S_1(T)-S_2(T)\right]=\\
&= E_Q\left[S_1(0)\exp\left(-\frac{\sigma_1^2 T}{2}+\sigma_1 W_1^Q(T)\right)\right]-E_Q\left[S_2(0)\exp\left(-\frac{\sigma_2^2 T}{2}+\sigma_2 W_2^Q(T)\right)\right]\\
&= S_1(0)-S_2(0).
\end{aligned}
$$

The corresponding variance is

$$
\sigma^2 = Var\left[S_1(T)-S_2(T)\right]=S_1^2(0)\exp\left(\sigma_1^2 T\right)+S_2^2(0)\exp\left(\sigma_2^2 T\right)-
$$
$$
2S_1(0)S_2(0)\exp\left[-\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right)T+\frac{\sigma_1^2 T+\sigma_2^2 T+2\sigma_1^2\sigma_2^2\rho T}{2}\right]-[S_1(0)-S_2(0)]^2.
$$

Let us call the approximation in equation (9) as a *normal approximation*.

The second spread option pricing formula that we are considering in the paper was proposed by Bjerksund and Stensland (2006), where the authors consider the following expectation:

$$
E_Q\left[\left(S_1(T)-S_2(T)-K\right)I_{\left\{S_1(T)\geq\frac{c(S_2(T))^2}{E_Q\left[(S_2(T))^2\right]}\right\}}\right], \tag{10}
$$

where

$$
\begin{aligned}
c &= S_2(0)+K,\\
b &= \frac{S_2(0)}{S_2(0)+K}.
\end{aligned}
$$

The strategy to exercise the option depends on the price of the long asset at maturity exceeding the power function of the short asset times a constant term. The price of the spread option is then given by

$$
p=S_1(0)\Phi(d_1)-S_2(0)\Phi(d_2)-K\Phi(d_3), \tag{11}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S_1(0)}{c}\right)+\left(\frac{\sigma_1^2}{2}-\sigma_1\sigma_2 b\rho+\frac{\sigma_2^2 b^2}{2}\right)T}{\sigma\sqrt{T}},\\
d_2 &= \frac{\ln\left(\frac{S_1(0)}{c}\right)+\left(\frac{\sigma_1^2}{2}+\sigma_1\sigma_2\rho+\frac{\sigma_2^2 b^2}{2}-\sigma_2^2 b\right)T}{\sigma\sqrt{T}},\\
d_3 &= \frac{\ln\left(\frac{S_1(0)}{c}\right)+\left(-\frac{\sigma_1^2}{2}+\frac{\sigma_2^2 b^2}{2}\right)T}{\sigma\sqrt{T}},\\
\sigma &= \sqrt{\sigma_1^2-2\sigma_1\sigma_2 b\rho+\sigma_2^2 b^2}.
\end{aligned}
$$

The authors showed, via numerical simulations, that equation (11) provides a very accurate lower bound to the true price of the contingent claim. It offers better estimates than the widely used Kirk's approximation (1995). Let us call the approximation as the *BS-approximation*. The derivation of equation (11) is in Appendix C.

To compare the two pricing formulas, we have estimated the prices by first varying the volatility of the first stock σ_1 and the time to maturity T parameters. The other parameters used are as follows: $S_1(0)=105, S_2(0)=100, K=5, \sigma_2=0.2, \rho=0.5$. The results we present in Table 2 (refer to Appendix B). Tables 3 and 4 show the absolute and percentage errors' values compared to Monte Carlo simulations. We can infer from the tables that the percentage errors vary significantly depending on the choice of market parameters for the proposed *normal approximation*, whereas the *BS-approximation* provides more accurate estimates. Lower rates of error are associated with a shorter time to maturity and the volatility parameters of the two stocks being closer to each other. Both pricing methodologies provide the lower bound on the option price compared to Monte Carlo simulations.

CVaR-hedging Methodology Adapted to Model (1)

Consider an $\mathcal{F}(T)$-measurable European style contingent claim $H \in L^1(Q)$, i.e. $E_Q(|H|)<\infty$, with the following payoff structure:

$$H=[S_1(T)-S_2(T)]^+. \tag{12}$$

Suppose that a financial institution has sold this option in the market and received $H(0)=E_Q(H)$, the amount required for perfect hedging, given by equation (6). However, the institution decides not to use all the proceeds from the sale of the option and thus is faced with the possibility of a shortfall at maturity. The question arises: What is the best trading strategy the institution should follow to minimise its expected shortfall if it uses CVaR as a measure of risk?

Denote by $\mathcal{A}$ the class of admissible self-financing trading strategies $\pi=(V(0),\xi,\eta)$, where $V(0)>0$ is the amount of initial capital, ξ and η denote the number of units of the first and second stocks held in portfolio, respectively, such that

$$V(t)=V(0)+\int_0^t \xi(s)dS_1(s)+\int_0^t \eta(s)dS_2(s), \forall t\in[0,T], P-a.s. \tag{13}$$

Strategy π is admissible if $V(t)\geq 0, \forall t\in[0,T], P-a.s.$

Denote by $\widehat{V}(0)$ the amount available for hedging such that $\widehat{V}(0)<H(0)$. Then the amount of shortfall $L(\pi)$ associated with a given portfolio π takes the following form:

$$L(\pi)=H-V(T)=H-V(0)-\int_0^T \xi(s)dS_1(s)-\int_0^T \eta(s)dS_2(s). \tag{14}$$

Fix a confidence level $\mathcal{L}$, usually 90%, 95% or 99% for practical purposes. We will be minimising $CVaR_{\mathcal{L}}$ over all strategies $\pi\in\mathcal{A}$ with the restriction on the amount of capital available, $V(0)\leq\widehat{V}(0)$, i.e.

$$\begin{cases} CVaR_{\mathcal{L}}(\pi)\to\min\limits_{\pi}, \\ \pi\in\mathcal{A}, V(0)\leq\widehat{V}(0). \end{cases} \tag{15}$$

Denote by $\mathcal{A}(\widehat{V}(0))$ the set of all admissible self-financing strategies that use no more initial capital than $\widehat{V}(0)$. Let us introduce an auxiliary function F as follows:

$$F_{\mathcal{L}}(\pi,z)=z+\frac{1}{1-\mathcal{L}}E[(L(\pi)-z)^+]. \tag{16}$$

According to Rockafellar and Urysev (2000), $CVaR_{\mathcal{L}}(\pi)$ and $F_{\mathcal{L}}(\pi,z)$ are interconnected through the following nice property: function $F_{\mathcal{L}}(\pi,z)$ is finite and convex with respect to $z\in\mathbb{R}$, and

$$CVaR_{\mathcal{L}}(\pi)=\min_{z\in\mathbb{R}}F_{\mathcal{L}}(\pi,z). \tag{17}$$

Moreover, minimising $CVaR_{\mathcal{L}}(\pi)$ over all strategies $\pi\in\mathcal{A}\left(\widehat{V}(0)\right)$ is equivalent to minimizing $F_{\mathcal{L}}(\pi,z)$ over all $(\pi,z)\in\mathcal{A}\left(\widehat{V}(0)\right)\times\mathbb{R}$:

$$\min_{\pi\in\mathcal{A}\left(\widehat{V}(0)\right)}CVaR_{\mathcal{L}}(\pi)=\min_{(\pi,z)\in\mathcal{A}\left(\widehat{V}(0)\right)\times\mathbb{R}}F_{\mathcal{L}}(\pi,z).$$

We arrive at the following equality:

$$\min_{\pi\in\mathcal{A}\left(\widehat{V}(0)\right)}CVaR_{\mathcal{L}}(\pi)=\min_{z\in\mathbb{R}}\left\{\min_{\pi\in\mathcal{A}\left(\widehat{V}(0)\right)}\left[z+\frac{1}{1-\mathcal{L}}E(H-V(T)-z)^{+}\right]\right\}. \tag{18}$$

The expression in equation (18) represents a new optimisation objective. Let us define an auxiliary function $c(z)$ in the following way:

$$c(z)=\min_{\pi\in\mathcal{A}\left(\widehat{V}(0)\right)}\left[z+\frac{1}{1-\mathcal{L}}E\left(H-V(T)-z\right)^{+}\right], \tag{19}$$

and rewrite equation (18) in terms of the new function $c(z)$ as follows:

$$\min_{\pi\in\mathcal{A}\left(\widehat{V}(0)\right)}CVaR_{\mathcal{L}}(\pi)=\min_{z\in\mathbb{R}}c(z). \tag{20}$$

Let the minimum value of the function $c(z)$ for each z be achieved using strategy

$$\tilde{\pi}(z)=\left(\tilde{V}(0,z),\tilde{\xi}(z),\tilde{\eta}(z)\right).$$

We then have:

$$\min_{\pi\in\mathcal{A}\left(\widehat{V}(0)\right)}E(H-V(T)-z)^{+}=E(H-\tilde{V}(T,z)-z)^{+},$$

where

$$\tilde{V}(T,z)=\tilde{V}(0,z)+\int_0^T\tilde{\xi}(s,z)dS_1(s)+\int_0^T\tilde{\eta}(s,z)dS_2(s)$$

Suppose that the global minimum of the function $c(z)$ is achieved at the point $\tilde{z}$, i.e.,

$$\min_{z\in\mathbb{R}}c(z)=c(\tilde{z}).$$

Then the optimal solution to the problem of $CVaR_{\mathcal{L}}$ minimization over all $\pi\in\mathcal{A}\left(\widehat{V}(0)\right)$, set in equation (15), is the strategy

$$\tilde{\pi}(\tilde{z})=\left\{\tilde{V}(0,\tilde{z}),\tilde{\xi}(\tilde{z}),\tilde{\eta}(\tilde{z})\right\}.$$

Now, according to equation (17), we have:

$$CVaR_{\mathcal{L}}(\tilde{\pi})=c(\tilde{z}). \tag{21}$$

It follows that if we can find the strategy $\tilde{\pi}$ in an explicit form, then the problem of $CVaR_{\mathcal{L}}$ minimisation will be reduced to the problem of minimisation of the function $c(z)$. Observe that for each z, the strategy $\tilde{\pi}$ is a solution to the following problem

$$E(H-V(T)-z)^{+} \to \min_{\pi\in\mathcal{A}(\widehat{V}(0))} \tag{22}$$

Let us note that

$$(H-V(T)-z)^{+} = [(H-z)^{+} - V(T)]^{+}.$$

Denote $(H-z)^{+}$ by $H(z)$. It is evident that $H(z)$ is an $\mathcal{F}$-measurable random variable, $H(z)\in L^{1}(Q)$ and $H(z)\geq 0$. We can consider $H(z)$ as a contingent claim. Equation (22) can be reformulated in the following form:

$$E(H(z)-V(T))^{+} \to \min_{\pi\in\mathcal{A}(\widehat{V}(0))} \tag{23}$$

This optimisation problem can be interpreted as the problem of expected shortfall minimisation over the strategy set $\mathcal{A}(\widehat{V}(0))$ of contingent claim $H(z)$, which was solved by Melnikov and Smirnov (2012). The optimal solution $\tilde{\pi}=(\tilde{V}(0),\tilde{\xi},\tilde{\eta})$ of the problem of expected shortfall minimisation is the perfect hedge of the modified contingent claim $\tilde{H}(z)=\tilde{\varphi}(z)(H-z)^{+}$ or, equivalently, $\tilde{H}(z)=\tilde{\varphi}(z)H(z)$:

$$E_Q\left(\tilde{H}(z)\,|\,\mathcal{F}(\mathrm{t})\right)=\tilde{V}(0,z)+\int_0^T\tilde{\xi}(s,z)dS_1(s)+\int_0^T\tilde{\eta}(s,z)dS_2(s),\forall t\in[0,T],P-a.s., \tag{24}$$

where

$$\tilde{\varphi}(z)=I_{\left\{\frac{dP}{dQ}>\tilde{a}(z)\right\}}+\beta(z)I_{\left\{\frac{dP}{dQ}=\tilde{a}(z)\right\}},$$

$$\tilde{a}(z)=\inf\left\{a\geq 0: E_Q\left[(H-z)^{+}I_{\left\{\frac{dP}{dQ}>a\right\}}\right]\leq\widehat{V}(0)\right\},$$

$$\beta(z)=\frac{\widehat{V}(0)-E_Q\left[(H-z)^{+}I_{\left\{\frac{dP}{dQ}>\tilde{a}(z)\right\}}\right]}{E_Q\left[(H-z)^{+}I_{\left\{\frac{dP}{dQ}=\tilde{a}(z)\right\}}\right]}.$$

Moreover, in the context of equation (21), the function $c(z)$ admits the following description:

$$c(z)=\begin{cases} z+\dfrac{1}{1-\mathcal{L}}E\left[(1-\tilde{\varphi}(z))(H-z)^{+}\right], z<\hat{z} \\ z, z\geq\hat{z}\end{cases} \tag{25}$$

Equivalently,

$$c(z)=\begin{cases} z+\dfrac{1}{1-\mathcal{L}}E\left[H(z)-\tilde{H}(z)\right], z<\hat{z} \\ z, z\geq\hat{z}\end{cases}$$

where $\hat{z}$ is the solution to the following equation:

$$\widehat{V}(0)=E_Q[(H-z)^{+}]. \tag{26}$$

3 Main Results: Extended Approximate Formulas

To find the price of the optimal hedge, in CVaR sense, or, equivalently, to construct a replicating portfolio with the lowest level of $CVaR_{\mathcal{L}}$, we must first find $\hat{z}$ in equation (26). To do it, we will be using the proposed *normal approximation,* and the *BS-approximation* described above. Once we have determined the unique value $\hat{z}$, we can minimise equation (25) numerically, using the Monte Carlo simulation technique. Suppose that $\tilde{z}$ is the global minimum of the function $c(z)$. Noting that the distribution of Brownian motion is atomless, the problem is reduced to evaluating the following expectation:

$$E_Q\left(\tilde{H}(\tilde{z})\right)=E_Q\left[\left(S_1(T)-S_2(T)-\tilde{z}\right)^+ I_{\left\{\frac{dP}{dQ}>\tilde{a}(\tilde{z})\right\}}\right]. \tag{27}$$

Depending on the chosen approximating method, the following two theorems provide the necessary tools for constructing a hedge with the lowest level of $CVaR_{\mathcal{L}}$:

Theorem 1. *Approximating the distribution of the difference between two log-normally distributed stock prices as a normal distribution, the price* p *of setting up a replicating portfolio for a spread option at any time* $t \le T$ *can be estimated as follows:*

$$p=S_1(0)\Phi^2\left(\hat{x}_1,\hat{y}_1,\rho_3\right)-S_2(0)\Phi^2\left(\hat{x}_2,\hat{y}_2,\rho_3\right)-\tilde{z}\Phi^2\left(\hat{x}_3,\hat{y}_3,\rho_3\right), \tag{28}$$

where

$$\hat{x}_1=\frac{m-\tilde{z}}{\sigma}+\sigma_1\rho_1\sqrt{T},$$

$$\hat{y}_1=\tilde{K}+\sigma_1\rho_4\sqrt{T},$$

$$\hat{x}_2=\frac{m-\tilde{z}}{\sigma}+\sigma_2\rho_2\sqrt{T},$$

$$\hat{y}_2=\tilde{K}+\sigma_2\rho_5\sqrt{T},$$

$$\hat{x}_3=\frac{m-\tilde{z}}{\sigma},$$

$$\hat{y}_3=\tilde{K},$$

$$\rho_1=\frac{\sqrt{T}}{\sigma}\left(S_1(0)\sigma_1-S_2(0)\sigma_2\rho\right),$$

$$\rho_2=\frac{\sqrt{T}}{\sigma}\left(S_1(0)\sigma_1\rho-S_2(0)\sigma_2\right),$$

$$\rho_3=\frac{a-b}{\sigma},$$

$$\rho_4=-\frac{\rho\phi_2+\phi_1}{\sigma_\phi},$$

$$\rho_5=-\frac{\phi_2+\phi_1\rho}{\sigma_\phi},$$

$$\sigma_\phi=\sqrt{\phi_1^2+\phi_2^2+2\rho\phi_1\phi_2},$$

$$a=S_2(0)\frac{\sigma_2\sqrt{T}}{\sigma_\phi}\left(\phi_1\rho+\phi_2\right),$$

$$b=S_1(0)\frac{\sigma_1\sqrt{T}}{\sigma_\phi}\left(\phi_1+\phi_2\rho\right),$$

$$\tilde{K}=\frac{\left(\frac{\sigma_\phi^2}{2}+\phi_1\theta_1+\phi_2\theta_2\right)T+\ln\left(\frac{1}{\tilde{a}(\tilde{z})}\right)}{\sigma_\phi\sqrt{T}},$$

$$S_1(T)-S_2(T)\approx\gamma\sim N\left(m,\sigma^2\right),$$

and $\Phi^2(x_1, x_2, \rho)$ *is the CDF of the bivariate standard normal distribution with correlation* ρ.

Proof. We want to find the price of the option with the following payoff

$$(S_1(T) - S_2(T) - K)^+ I_{\left\{\frac{dP}{dQ} > \tilde{a}(\tilde{z})\right\}}.$$

Let us first consider the expression in the indicator function:

$$\begin{aligned}
\left\{\frac{dP}{dQ} > \tilde{a}(\tilde{z})\right\} &= \left\{\frac{1}{\tilde{a}(\tilde{z})} > \exp\left[\phi_1 W_1^Q(T) + \phi_2 W_2^Q(T) - \left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T\right]\right\} \\
&= \left\{\frac{\phi_1 W_1^Q(T) + \phi_2 W_2^Q(T)}{\sigma_\phi\sqrt{T}} < \frac{\left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T + \ln\left(\frac{1}{\tilde{a}(\tilde{z})}\right)}{\sigma_\phi\sqrt{T}}\right\} \\
&= \left\{\in < \tilde{K}\right\},
\end{aligned} \tag{29}$$

where

$$\in = \frac{\tilde{W}^Q(T)}{\sqrt{T}} \sim N(0,1),$$

$$\tilde{W}^Q(T) = \frac{\phi_1 W_1^Q(T) + \phi_2 W_2^Q(T)}{\sigma_\phi},$$

$$\tilde{K} = \frac{\left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T + \ln\left(\frac{1}{\tilde{a}(\tilde{z})}\right)}{\sigma_\phi\sqrt{T}},$$

$$\phi_1 = \frac{\sigma_1\mu_2\rho - \sigma_2\mu_1}{\sigma_1\sigma_2(1-\rho^2)},$$

$$\phi_2 = \frac{\sigma_2\mu_1\rho - \sigma_1\mu_2}{\sigma_1\sigma_2(1-\rho^2)},$$

$$\sigma_\phi = \sqrt{\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2\rho},$$

$$\theta_1 = \frac{\mu_1}{\sigma_1},$$

$$\theta_2 = \frac{\mu_2}{\sigma_2}.$$

Replacing $S_1(T) - S_2(T)$ by $\gamma \sim N(m, \sigma^2)$, we obtain:

$$\begin{aligned}
&E_Q\left[\left(S_1(T)-S_2(T)-K\right)^+ I_{\left\{\frac{dP}{dQ}>\tilde{a}(\tilde{z})\right\}}\right] \\
&= E_Q\left[\left(S_1(T)-S_2(T)-K\right) I_{\{S_1(T)-S_2(T)>K\}} I_{\left\{\frac{dP}{dQ}>\tilde{a}(\tilde{z})\right\}}\right] \\
&= E_Q\left[S_1(T) I_{\{\gamma>K\}} I_{\{\in<\tilde{K}\}}\right] - E_Q\left[S_2(T) I_{\{\gamma>K\}} I_{\{\in<\tilde{K}\}}\right] - KE_Q\left[I_{\{\gamma>K\}} I_{\{\in<\tilde{K}\}}\right] \\
&= S_1(0)\exp\left(-\frac{\sigma_1^2 T}{2}\right) E_Q\left[\exp\left(\sigma_1 W_1^Q(T)\right) I_{\{-\gamma<-K\}} I_{\{\in<\tilde{K}\}}\right] \\
&-S_2(0)\exp\left(-\frac{\sigma_2^2 T}{2}\right) E_Q\left[\exp\left(\sigma_2 W_2^Q(T)\right) I_{\{-\gamma<-K\}} I_{\{\in<\tilde{K}\}}\right] - KE_Q\left[I_{\{-\gamma<-K\}} I_{\{\in<\tilde{K}\}}\right].
\end{aligned} \tag{30}$$

Consider the first term in equation (30):

$$\begin{aligned}
&S_1(0)\exp\left(-\frac{\sigma_1^2 T}{2}\right) E_Q\left[\exp\left(\sigma_1 W_1^Q(T)\right) I_{\{-\gamma<-K\}} I_{\{\grave{o}<\tilde{K}\}}\right] = \\
&S_1(0)\exp\left(-\frac{\sigma_1^2 T}{2}\right) E_Q\left[\exp\left(-Z_1\right) I_{\{X<-K\}} I_{\{Y<\tilde{K}\}}\right],
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
Z_1 &= -\sigma_1 W_1^Q(T) \sim N\left(0, \sigma_1^2 T\right), \\
X &= -\gamma \sim N\left(-m, \sigma^2\right), \\
Y &= \frac{\phi_1 W_1^Q(T) + \phi_2 W_2^Q(T)}{\sigma_\phi \sqrt{T}} \sim N(0,1).
\end{aligned}$$

To apply the *two-asset lemma* (see Appendix A) to the expectation term, we need to estimate the correlation coefficients ρ_{Z_1X}, ρ_{Z_1Y} and ρ_{XY}. Consider ρ_{XY},

$$\rho_{XY} = \frac{\sigma_{xy}^2}{\sigma_x \sigma_y}.$$

Since $Y \sim N(0,1)$,

$$\begin{aligned}
\sigma_{xy}^2 &= E_Q(XY) \\
&= E_Q\left[\frac{\phi_1 W_1^Q(T) + \phi_2 W_2^Q(T)}{\sigma_\phi \sqrt{T}} S_2(T)\right] - E_Q\left[\frac{\phi_1 W_1^Q(T) + \phi_2 W_2^Q(T)}{\sigma_\phi \sqrt{T}} S_1(T)\right] \\
&= S_2(0)\exp\left(-\frac{\sigma_2^2 T}{2}\right)\frac{1}{\sigma_\phi \sqrt{T}} E_Q\left[\phi_1 W_1^Q(T)\exp\left(\sigma_2 W_2^Q(T)\right) + \phi_2 W_2^Q(T)\exp\left(\sigma_2 W_2^Q(T)\right)\right] \\
&-S_1(0)\exp\left(-\frac{\sigma_1^2 T}{2}\right)\frac{1}{\sigma_\phi \sqrt{T}} E_Q\left[\phi_1 W_1^Q(T)\exp\left(\sigma_1 W_1^Q(T)\right) + \phi_2 W_2^Q(T)\exp\left(\sigma_1 W_1^Q(T)\right)\right]
\end{aligned}$$

Opening the brackets and calculating the expectations, the above yields:

$$S_2(0)\frac{\sigma_2\sqrt{T}}{\sigma_\phi}(\phi_1\rho+\phi_2)-S_1(0)\frac{\sigma_1\sqrt{T}}{\sigma_\phi}(\phi_1+\phi_2\rho).$$

To get the correlation, we need to divide it by $\sigma_x\sigma_y$ to get:

$$\rho_{XY}=\frac{a-b}{\sigma},$$

where

$$\begin{aligned} a &= S_2(0)\frac{\sigma_2\sqrt{T}}{\sigma_\phi}(\phi_1\rho+\phi_2), \\ b &= S_1(0)\frac{\sigma_1\sqrt{T}}{\sigma_\phi}(\phi_1+\phi_2\rho). \end{aligned}$$

Now consider ρ_{Z_1X},

$$\rho_{Z_1X}=\frac{\sigma^2_{z_1x}}{\sigma_{z_1}\sigma_y}.$$

Since $Z_1\sim N\left(0,\sigma_1^2T\right)$,

$$\begin{aligned} \sigma^2_{z_1x} &= E_Q(Z_1X) \\ &= E_Q\left\{\sigma_1W_1^Q(T)\left[S_1(0)\exp\left(-\frac{\sigma_1^2T}{2}+\sigma_1W_1^Q(T)\right)-S_2(0)\exp\left(-\frac{\sigma_2^2T}{2}+\sigma_2W_2^Q(T)\right)\right]\right\} \\ &= S_1(0)\exp\left(-\frac{\sigma_1^2T}{2}\right)E_Q\left[\sigma_1W_1^Q(T)\exp\left(\sigma_1W_1^Q(T)\right)\right] \\ &\quad -S_2(0)\exp\left(-\frac{\sigma_2^2T}{2}\right)E_Q\left[\sigma_1W_1^Q(T)\exp\left(\sigma_2W_2^Q(T)\right)\right]. \end{aligned}$$

Opening the brackets and calculating the expectations,

$$\sigma^2_{z_1x}=S_1(0)\sigma_1^2T-S_2(0)\sigma_1\sigma_2\rho T.$$

Similarly, by dividing by $\sigma_x\sigma_y$ we get the correlation:

$$\rho_{Z_1X}=\frac{\sqrt{T}}{\sigma}\left(S_1(0)\sigma_1-S_2(0)\sigma_2\rho\right).$$

Let us now consider ρ_{Z_1Y},

$$\rho_{Z_1Y}=\frac{\sigma^2_{z_1y}}{\sigma_{z_1}\sigma_y}.$$

Since both random variables Z_1 and Y have zero expectation,

$$\sigma^2_{z_1 y} = E_Q\left(Z_1 Y\right)$$
$$= E_Q\left[-\sigma_1 W_1^Q\left(T\right)\left(\frac{\phi_1 W_1^Q\left(T\right)+\phi_2 W_2^Q\left(T\right)}{\sigma_\phi\sqrt{T}}\right)\right].$$

Simplifying, we get:

$$\rho_{Z_1 Y} = -\frac{\phi_1 + \rho\phi_2}{\sigma_\phi}.$$

We can now apply the *two-asset lemma* to equation (31) to get

$$S_1\left(0\right)\exp\left(-\frac{\sigma_1^2 T}{2}\right)E_Q\left[\exp\left(-Z_1\right)I_{\{X<-K\}}I_{\{Y<\tilde{K}\}}\right] \\ = S_1\left(0\right)\Phi^2\left(\hat{x}_1,\hat{y}_1,\rho_{XY}\right), \tag{32}$$

where

$$\hat{x}_1 = \frac{m-K}{\sigma} + \sigma_1\rho_{Z_1 X}\sqrt{T}, \quad \hat{y}_1 = \tilde{K} + \sigma_1\sqrt{T}\rho_{Z_1 Y}, \quad \rho_{Z_1 X} = \frac{\sqrt{T}}{\sigma}\left(S_1\left(0\right)\sigma_1 - S_2\left(0\right)\sigma_2\rho\right), \quad \rho_{Z_1 Y} = -\frac{\phi_1 + \phi_2\rho}{\sigma_\rho},$$

$$\rho_{XY} = \frac{a-b}{\sigma},\ a = S_2\left(0\right)\frac{\sigma_2\sqrt{T}}{\sigma_\phi}\left(\phi_1\rho + \phi_2\right),\ b = S_1\left(0\right)\frac{\sigma_1\sqrt{T}}{\sigma_\phi}\left(\phi_1 + \phi_2\rho\right),\ \tilde{K} = \frac{\left(\frac{\sigma_\phi^2}{2} + \phi_1\theta_1 + \phi_2\theta_2\right)T + \ln\left(\frac{1}{\tilde{a}\left(\tilde{z}\right)}\right)}{\sigma_\phi\sqrt{T}}$$

Now consider the second term in equation (30):

$$S_2\left(0\right)\exp\left(-\frac{\sigma_2^2 T}{2}\right)E_Q\left[\exp\left(\sigma_2 W_2^Q\left(T\right)\right)I_{\{-\gamma<-K\}}I_{\{\epsilon<\tilde{K}\}}\right], \tag{33}$$

where

$$Z_2 = -\sigma_2 W_2^Q\left(T\right) \sim N\left(0,\sigma_1^2 T\right),\ X = -\gamma \sim N\left(-m,\sigma^2\right),\ Y = \frac{\phi_1 W_1^Q\left(T\right)+\phi_2 W_2^Q\left(T\right)}{\sigma\_\phi\sqrt{T}} \sim N\left(0,1\right).$$

We need to estimate the correlation coefficients $\rho_{Z_2 X}$ and $\rho_{Z_2 Y}$. Proceeding in the same manner as for equation (31), we evaluate the correlation coefficients to be as follows:

$$\rho_{Z_2 X} = \frac{\sqrt{T}}{\sigma}\left(S_1\left(0\right)\sigma_1\rho - S_2\left(0\right)\sigma_2\right),$$
$$\rho_{Z_2 Y} = -\frac{\phi_1\rho + \phi_2}{\sigma_\phi}.$$

Applying the *two-asset lemma*,

$$S_2\left(0\right)\exp\left(-\frac{\sigma_2^2 T}{2}\right)E_Q\left[\exp\left(-Z_2\right)I_{\{X<-K\}}I_{\{Y<\tilde{K}\}}\right] \\ = S_2\left(0\right)\Phi^2\left(\hat{x}_2,\hat{y}_2,\rho_{XY}\right), \tag{34}$$

where

$$\hat{x}_2 = \frac{m-K}{\sigma} + \sigma_2\rho_{Z_2X}\sqrt{T},\ \hat{y}_2 = \tilde{K} + \sigma_2\sqrt{T}\rho Z_{2Y},\ \rho_{Z_2X} = \frac{\sqrt{T}}{\sigma}\left(S_1(0)\sigma_1\rho - S_2(0)\sigma_2\right),\ \rho_{Z_2Y} = -\frac{\phi_1\rho+\phi_2}{\sigma_\phi}.$$

The last term in equation (30) is simply

$$KE_Q\left[I_{\{-\gamma<-K\}}I_{\{\epsilon<\tilde{K}\}}\right] = K\Phi^2\left(\hat{x}_3, \hat{y}_3, \rho_{XY}\right), \tag{35}$$

where

$$\hat{x}_3 = \frac{m-K}{\sigma},\ \hat{y}_3 = \tilde{K}.$$

Combining all three terms, equations (32), (34) and (35), we get the stated formula in equation (28).

Alternatively, using the *BS-approximation*:

Theorem 2. *Using the BS-approximation for the price of the spread option, the price p of setting up the replicating portfolio at any time $t \le T$ can be estimated as follows:*

$$p = S_1(0)\Phi^2\left(\hat{x}_1, \hat{y}_1, \rho_3\right) - S_2(0)\Phi^2\left(\hat{x}_2, \hat{y}_2, \rho_3\right) - \tilde{z}\Phi^2\left(\hat{x}_3, \hat{y}_3, \rho_3\right), \tag{36}$$

where

$$\hat{x}_1 = \widehat{K} + \sigma_1\rho_1\sqrt{T},\quad \hat{y}_1 = \tilde{K} + \sigma_1\rho_4\sqrt{T},\quad \hat{x}_2 = \widehat{K} + \sigma_2\rho_2\sqrt{T},\quad \hat{y}_2 = \tilde{K} + \sigma_2\rho_5\sqrt{T},\quad \hat{x}_3 = \widehat{K},\quad \hat{y}_3 = \tilde{K},$$

$$\rho_1 = \frac{\left(\sigma_1 - \sigma_2 b\rho\right)\sqrt{T}}{\sqrt{\sigma_1^2 T - 2\sigma_1\sigma_2 b\rho T + \sigma_2^2 b^2 T}}, \rho_2 = \frac{\left(\sigma_1\rho - \sigma_2 b\right)\sqrt{T}}{\sqrt{\sigma_1^2 T - 2\sigma_1\sigma_2 b\rho T + \sigma_2^2 b^2 T}}, \rho_3 = \frac{\left(\sigma_2 b\phi_1\rho + \sigma_2 b\phi_2 - \sigma_1\phi_1 - \sigma_1\phi_2\rho\right)\sqrt{T}}{\sigma_\phi\sqrt{\sigma_1^2 T + \sigma_2^2 b^2 T - 2\sigma_1\sigma_2 b\rho T}},$$

$$\rho_4 = -\frac{\phi_2\rho + \phi_1}{\sigma_\phi},\qquad \rho_5 = -\frac{\phi_2 + \phi_1\rho}{\sigma_\phi},\qquad \sigma_\phi = \sqrt{\phi_1^2 + \phi_2^2 + 2\rho\phi_1\phi_2},\qquad b = \frac{S_2(0)}{c},\qquad c = S_2(0) + \tilde{z},$$

$$\tilde{K} = \frac{\left(\frac{\sigma_\phi^2}{2} + \sigma_1\phi_1 + \sigma_2\phi_2\right) + \ln\left(\frac{1}{\tilde{a}(\tilde{z})}\right)}{\sigma_\phi\sqrt{T}},\ \widehat{K} = \frac{\ln\left(\frac{S_1(0)}{a(\tilde{z})}\right) - \frac{\sigma_1^2 T}{2} + \frac{\sigma_2^2 b^2 T}{2}}{\sqrt{\sigma_1^2 T + \sigma_2^2 b^2 T - 2\rho\sigma_1\sigma_2 b T}}.$$

Proof. We need to estimate the following expectation:

$$E_Q\left[\left(S_1(T) - S_2(T) - K\right)I_{\left\{\frac{dP}{dQ}>\tilde{a}(\tilde{z})\right\}}I_{\left\{S_1(T)\ge\frac{c\left(S_2(T)\right)^b}{E_Q\left(\left(S_2(T)\right)^b\right)}\right\}}\right].$$

The first indicator function has already been considered in equation (29); the term in the second indicator function was considered in equation (45) of Appendix C. We can rewrite the above expectation in the following way:

$$E_Q\left[S_1(T)I_{\{\epsilon_1\le\tilde{K}\}}I_{\{\epsilon_2\le\overline{K}\}}\right]-E_Q\left[S_2(T)I_{\{\epsilon_1\le\tilde{K}\}}I_{\{\epsilon_2\le\overline{K}\}}\right]-KE_Q\left[I_{\{\epsilon_1\le\tilde{K}\}}I_{\{\epsilon_2\le\overline{K}\}}\right]. \tag{37}$$

We can apply the *two-asset lemma* to each of the three terms in equation (37). Before that, however, we need to estimate the correlation coefficient between ϵ_1 and ϵ_2,

$$\rho_{\epsilon_1\epsilon_2}=E_Q\left[\left(\frac{\phi_1W_1^Q(T)+\phi_2W_2^Q(T)}{\sigma_\phi\sqrt{T}}\right)\left(\frac{\sigma_2bW_2^Q(T)-\sigma_1W_1^Q(T)}{\sqrt{\sigma_1^2T-2\sigma_1\sigma_2b\rho T+\sigma_2^2b^2T}}\right)\right]=$$

$$=\frac{(\sigma_2b\phi_1\rho+\sigma_2b\phi_2-\sigma_1\phi_1-\sigma_1\phi_2\rho)\sqrt{T}}{\sigma_\phi\sqrt{\sigma_1^2T+\sigma_2^2b^2T-2\sigma_1\sigma_2b\rho T}},$$

where we used the fact that

$$E\left[W^2(T)\right]=Var\left[W(T)\right]=T,$$

and

$$E\left[W_1(T)W_2(T)\right]=Cov\left[W_1(T)W_2(T)\right]=\rho T.$$

Combining this result with the results of equations (29) and (45), and applying the *two-asset lemma* to the first term of equation (37),

$$E_Q\left[S_1(T)I_{\{\epsilon_1\le\tilde{K}\}}I_{\{\epsilon_2\le\overline{K}\}}\right]=S_1(0)\Phi^2\left(\hat{x}_1,\hat{y}_1,\rho_3\right), \tag{38}$$

where

$$\hat{x}_1=\tilde{K}+\sigma_1\rho_1\sqrt{T},$$

$$\hat{y}_1=\overline{K}+\sigma_1\rho_4\sqrt{T},$$

$$\rho_1=\frac{(\sigma_1-\sigma_2b\rho)\sqrt{T}}{\sqrt{\sigma_1^2T-2\sigma_1\sigma_2b\rho T+\sigma_2^2b^2T}},$$

$$\rho_3=\frac{(\sigma_2b\phi_1\rho+\sigma_2b\phi_2-\sigma_1\phi_1-\sigma_1\phi_2\rho)\sqrt{T}}{\sigma_\phi\sqrt{\sigma_1^2T+\sigma_2^2b^2T-2\sigma_1\sigma_2b\rho T}},$$

$$\rho_4=-\frac{\phi_2\rho+\phi_1}{\sigma_\phi}.$$

The second term of equation (37) evaluates to

$$E_Q\left[S_2(T)I_{\{\epsilon_1\le\tilde{K}\}}I_{\{\epsilon_2\le\overline{K}\}}\right]=S_2(0)\Phi^2\left(\hat{x}_2,\hat{y}_2,\rho_3\right), \tag{39}$$

where

$$\hat{x}_2 = \tilde{K} + \sigma_2\rho_2\sqrt{T},$$

$$\hat{y}_2 = \overline{K} + \sigma_2\rho_5\sqrt{T},$$

$$\rho_2 = \frac{(\sigma_1\rho - \sigma_2 b)\sqrt{T}}{\sqrt{\sigma_1^2 T - 2\sigma_1\sigma_2 b\rho T + \sigma_2^2 b^2 T}},$$

$$\rho_5 = -\frac{\phi_2 + \phi_1\rho}{\sigma_\phi}.$$

The last term of equation (37) is

$$KE_Q\left[I_{\{\epsilon_1 \le \tilde{K}\}} I_{\{\epsilon_2 \le \overline{K}\}}\right] = K\Phi\left(\hat{x}_3, \hat{y}_3, \rho_3\right), \tag{40}$$

where

$$\hat{x}_3 = \tilde{K},$$

$$\hat{y}_3 = \overline{K}.$$

Combining equations (38), (39) and (40), we get the formula stated in equation (36).

The existence of closed-form formulas for estimating CvaR-optimal option prices, as per Theorems 1 and 2, allows constant rebalancing of the replicating portfolio at any moment in time $t \le T$, which is vital for risk management purposes.

5 Numerical Illustration and Application to Regulatory Needs

To see how the methodology would apply to the real market data, we have downloaded the closing price data for Apple Inc. and S&P500 index from 1st January 2013 to 28th March 2018, with overall 1319 observations. Having transformed the prices to logarithmic returns and having annualised the returns, we obtained the following standard deviations: $\sigma_1 = 0.24, \sigma_2 = 0.12$, where subscript 1 refers

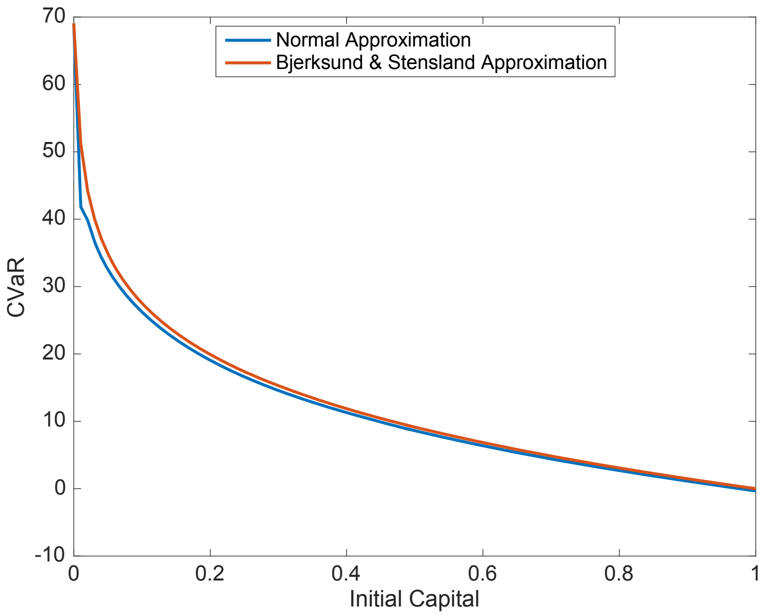

Figure 1. CVaR for varying level of initial capital at 99% confidence level

Source: The authors.

Table 1
CVaR at 99% confidence level

Capital Available, %	CVaR	
	Normal approximation	**BS-approximation**
0	68.9700	69.0788
10	26.0501	27.4378
20	18.9578	19.8209
30	14.4591	15.1719
40	11.1611	11.7549
51	8.4916	9.0196
61	6.2363	6.7189
71	4.2763	4.7194
81	2.5743	2.9413
91	0.9873	1.3326
100	0.0000	0.0000

Source: The authors.

to Apple Inc. and subscript 2 to S&P 500 index. The annualised returns are: $\mu_1 = 0.14, \mu_2 = 0.11$. An investor expects to earn a higher rate of return on Apple Inc. to compensate for higher volatility. The estimated correlation coefficient over the period was $\rho = 0.5068$. We have standardised the initial prices to be equal

$$S_1(0) = S_2(0) = 78.4329$$

The institution has sold an option to exchange a single unit of S&P500 for the unit of stock of Apple Inc. with an expiration date of one year from now. The price required for complete hedging is determined via equation (6) to be equal to $p = 6.49$. We estimate CVaR at 99%. Refer to Fig. 1, where we plot the level of $CVaR_{0.99}$ for varying levels of the initial capital available as a percentage of the arbitrage-free price.

Table 1 summarises the results of the simulation. We can see that the *normal approximation* underestimates CVaR for all levels of initial capital available. It is an expected result given that the *normal approximation* provides lower price estimates when compared with the *BS-approximation*. We note that both approaches offer only an approximation to the true level of CVaR because there is no exact pricing formula for equation (5).

We can further supplement our analysis by looking at CVaR-efficient portfolios from a regulatory point of view. Suppose that a regulator in the market requires the member institutions to keep a

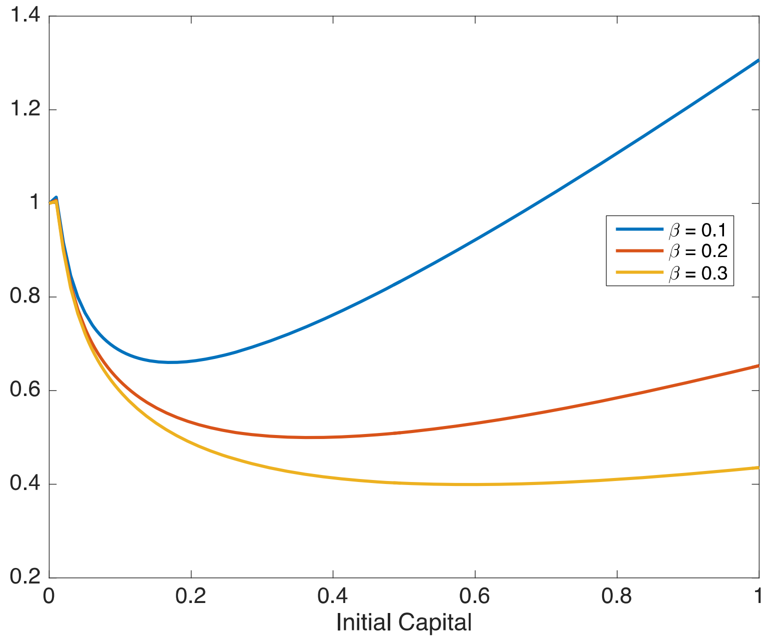

Figure 2. The relative attractiveness of CVaR-efficient portfolio at 99% confidence level

Source: The authors.

certain amount of capital in reserves depending on the estimated level of CVaR. Let β be the necessary amount of capital per unit of CVaR exposure. Denote by

$$\lambda_{\mathcal{L}}\left(\widehat{V}(0)\right)=\beta CVaR_{\mathcal{L}}\left(\widehat{V}(0)\right)+\widehat{V}(0) \tag{41}$$

the total amount of capital to be kept in reserves provided that the amount of $\widehat{V}(0)$ has been used for hedging purposes at the significance level $\mathcal{L}$. Then the $CVaR_{\mathcal{L}}$ of an unhedged position is $\lambda_{\mathcal{L}}(0)$. Introduce the following ratio:

$$\Theta_{\mathcal{L}}=\frac{\lambda_{\mathcal{L}}\left(\widehat{V}(0)\right)}{\lambda_{\mathcal{L}}(0)}. \tag{42}$$

The ratio tells us the relative attractiveness of a CVaR-efficient portfolio. Where $\Theta_{\mathcal{L}}<1$, engaging in CVaR-efficient hedging allows the institution to use less capital to meet the regulatory requirement as compared to an unhedged position and vice versa. We apply this line of analysis to our Apple Inc. and S&P500 portfolio at a 99% significance level, and the results we show in Fig. 1.

The above figure clearly indicates that the higher the regulatory requirements, the more attractive a CVaR-efficient portfolio is compared to a portfolio with no hedging. Also, the graph of the relative attractiveness of the CVaR-efficient portfolio as a function of the level of initial capital used is U-shaped, meaning that the relative effectiveness is more sensitive to changes in the capital employed in the tails of the graph. The reader can clearly see this effect from Table 1. The concavity of the graph in the markets with regulatory requirements means that we can optimise concerning the amount of initial capital to be used to maximise the replicating portfolio's effectiveness.

6 Conclusion

In this paper, we have investigated the problem of constructing CVaR-efficient portfolios under capital constraints in the Margrabe market model setting. The two different spread option pricing formulas used provided comparable results. However, neither of the two methods provides an exact solution since no closed form PDF for the difference between two log-normal random variables exists to this moment.

Acknowledgments

The research was supported by the NSERC Discovery Grants #5901 and RES 0043487.

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APPENDIXES

Appendix A Two-asset lemma

Lemma 1. *Let* $X \sim N\left(\mu_x, \sigma_x^2\right), Y \sim N\left(\mu_y, \sigma_y\right)$ *and* $Z \sim N\left(\mu_z, \sigma_z^2\right)$ *be three normally distributed random variables with correlations* $\rho_{XY}, \rho_{XZ}, \rho_{YZ}$. *Then,*

$$E\left[\exp(-Z) I_{\{X<x\}} I_{\{Y<y\}}\right] = \exp\left(-\mu_z + \frac{\sigma_z^2}{2}\right)\Phi^2\left(\hat{x}, \hat{y}, \rho_{XY}\right), \tag{43}$$

where

$$\hat{x} = \frac{x-\mu_x}{\sigma_x} + \sigma_z \rho_{XZ}, \ \hat{y} = \frac{y-\mu_y}{\sigma_y} + \sigma_z \rho_{YZ}$$

and Φ^2 *denotes the two-dimensional normal cumulative distribution function (see Melnikov (2011)).*

Appendix B Comparison results for normal approximation and BS-approximation

Table 2
Spread option: value approximation. The different formulas are from top to bottom: Monte-Carlo simulation, BS-approximation, the normal approximation

σ_1	T			
	0.5	**1**	**3**	**5**
	4.8886	6.9092	11.937	15.372
0.1	4.885	6.9041	11.928	15.361
	4.8399	6.8061	11.404	14.228
	5.1504	7.2788	12.573	16.188
0.15	5.1447	7.2708	12.559	16.17
	5.1124	7.1577	12.103	15.121
	5.7911	8.1831	14.127	18.179
0.2	5.7833	8.172	14.108	18.153
	5.7503	8.0437	13.445	16.897
	6.7008	9.4663	16.327	20.99
0.25	6.6925	9.4544	16.305	20.959
	6.6062	9.2315	15.358	18.908
	7.7852	10.994	18.937	24.313
0.3	7.7771	10.983	18.913	24.277
	7.7153	10.651	17.074	20.852

Source: The authors.

Table 3
Spread option: absolute error

σ_1	T			
	0.5	**1**	**3**	**5**
	0	0	0	0
0.1	−0.0036	−0.0051	−0.009	−0.011
	−0.0487	−0.1031	−0.533	−1.144
	0	0	0	0
0.15	−0.0057	−0.008	−0.014	−0.018
	−0.038	−0.1211	−0.47	−1.067
	0	0	0	0
0.2	−0.0078	−0.0111	−0.019	−0.026
	−0.0408	−0.1394	−0.682	−1.282
	0	0	0	0
0.25	−0.0083	−0.0119	−0.022	−0.031
	−0.0946	−0.2348	−0.969	−2.082
	0	0	0	0
0.3	−0.0081	−0.011	−0.024	−0.036
	−0.0699	−0.343	−1.863	−3.461

Source: The authors.

Table 4
Spread option: percentage error

y	T			
	0.5	**1**	**3**	**5**
	0.00%	0.00%	0.00%	0.00%
0.1	−0.07%	−0.07%	−0.08%	−0.07%
	−1.00%	−1.49%	−4.47%	−7.44%
	0.00%	0.00%	0.00%	0.00%
0.15	−0.11%	−0.11%	−0.11%	−0.11%
	−0.74%	−1.66%	−3.74%	−6.59%
	0.00%	0.00%	0.00%	0.00%
0.2	−0.13%	−0.14%	−0.13%	−0.14%
	−0.70%	−1.70%	−4.83%	−7.05%
	0.00%	0.00%	0.00%	0.00%
0.25	−0.12%	−0.13%	−0.13%	−0.15%
	−1.41%	−2.48%	−5.93%	−9.92%
	0.00%	0.00%	0.00%	0.00%
0.3	−0.10%	−0.10%	−0.13%	−0.15%
	−0.90%	−3.12%	−9.84%	−14.24%

Source: The authors.

Appendix C BS-approximation

Consider the following expression:

$$
\begin{aligned}
&E_Q\left[\left(S_1(T)-S_2(T)-K\right)I_{\left\{S_1(T)\geq\frac{c(S_2(T))^b}{E_Q\left((S_2(T))^b\right)}\right\}}\right]\\
&=E_Q\left[S_1(T)I_{\left\{S_1(T)\geq\frac{c(S_2(T))^b}{E_Q\left((S_2(T))^b\right)}\right\}}\right]-E_Q\left[S_2(T)I_{\left\{S_1(T)\geq\frac{c(S_2(T))^b}{E_Q\left((S_2(T))^b\right)}\right\}}\right]\\
&-E_Q\left[KI_{\left\{S_1(T)\geq\frac{c(S_2(T))^b}{E_Q\left((S_2(T))^b\right)}\right\}}\right].
\end{aligned}
\tag{44}
$$

The term in the denominator is

$$
E_Q\left(\left(S_2(T)\right)^b\right)=\left(S_2(0)\right)^b\exp\left(-\frac{\sigma_2^2bT}{2}\right)E_Q\left[\exp\left(\sigma_2bW_2^Q(T)\right)\right]=\left(S_2(0)\right)^b\exp\left(\frac{\sigma_2^2b(b-1)T}{2}\right).
$$

Let us now simplify the term in the indicator function:

$$
\begin{aligned}
&S_1(T)\geq\frac{c\left(S_2(T)\right)^b}{E_Q\left(\left(S_2(T)\right)^b\right)},\ S_1(0)\exp\left(-\frac{\sigma_1^2T}{2}+\sigma_1W_1^Q(T)\right)\geq c\exp\left(-\frac{\sigma_2^2b^2T}{2}+\sigma_2bW_2^Q(T)\right),\\
&\ln\left(\frac{S_1(0)}{c}\right)-\frac{\sigma_1^2T}{2}+\frac{\sigma_2^2b^2T}{2}\geq\sigma_2bW_2^Q(T)-\sigma_1W_1^Q(T).
\end{aligned}
\tag{45}
$$

Since

$$
\sigma_2bW_2^Q(T)-\sigma_1W_1^Q(T)\sim N\left(0,\sigma_1^2T-2\sigma_1\sigma_2b\rho T+\sigma_2^2b^2T\right),
$$

the inequality in equation (45) is equivalent to $\in\leq d_3$,

where

$$
\in=\frac{\sigma_2bW_2^Q(T)-\sigma_1W_1^Q(T)}{\sqrt{\sigma_1^2T-2\sigma_1\sigma_2b\rho T+\sigma_2^2b^2T}}\sim N(0,1),\ d_3=\frac{\ln\left(\frac{S_1(0)}{c}\right)-\frac{\sigma_1^2T}{2}+\frac{\sigma_2^2b^2T}{2}}{\sqrt{\sigma_1^2T-2\sigma_1\sigma_2b\rho T+\sigma_2^2b^2T}}.
$$

Consider the first term in the original expectation, i.e. equation (44),

$$E_Q\left[S_1(T)I_{\left\{S_1(T)\geq\frac{c(S_2(T))^b}{E_Q\left((S_2(T))^b\right)}\right\}}\right]=S_1(0)\exp\left(-\frac{\sigma_1^2T}{2}\right)E_Q\left[\exp\left(\sigma_1W_1^Q(T)\right)I_{\{\in\leq d_3\}}\right] \tag{46}$$

Applying the *two-asset lemma* to the expectation term,

$$E_Q\left[\exp\left(\sigma_1W_1^Q(T)\right)I_{\{\in\leq d_3\}}\right]=\exp\left(\frac{\sigma_1^2T}{2}\right)\Phi\left(d_3+\frac{(\sigma_1-\sigma_2b\rho)\sqrt{T}}{\sqrt{\sigma_1^2T-2\sigma_1\sigma_2b\rho T+\sigma_2^2b^2T}}\right),$$

which leads to

$$E_Q\left[S_1(T)I_{\left\{S_1(T)\geq\frac{c(S_2(T))^b}{E_Q\left((S_2(T))^b\right)}\right\}}\right]=S_1(0)\Phi(d_1),$$

where

$$d_1=d_3+\sigma_1\rho_1\sqrt{T},\ \rho_1=\frac{(\sigma_1-\sigma_2b\rho)\sqrt{T}}{\sqrt{\sigma_1^2T-2\sigma_1\sigma_2b\rho T+\sigma_2^2b^2T}}.$$

Let us consider the second term of equation (44),

$$E_Q\left[S_2(T)I_{\left\{S_1(T)\geq\frac{c(S_2(T))^b}{E_Q\left((S_2(T))^b\right)}\right\}}\right]=S_2(0)\exp\left(-\frac{\sigma_2^2T}{2}\right)E_Q\left[\exp\left(\sigma_2W_2^Q(T)\right)I_{\{\in\leq d_3\}}\right].$$

Applying the *two-asset lemma* again, we have

$$E_Q\left[S_2(T)I_{\left\{S_1(T)\geq\frac{cS_2^b(T)}{E_Q\left(S_2^b(T)\right)}\right\}}\right]=S_2(0)\Phi(d_2), \tag{47}$$

where

$$d_2=d_3+\sigma_2\rho_2\sqrt{T},\ \rho_2=\frac{(\sigma_1\rho-\sigma_2b)\sqrt{T}}{\sqrt{\sigma_1^2T-2\sigma_1\sigma_2b\rho T+\sigma_2^2b^2T}}.$$

Finally, the third term of equation (44) is simply

$$E_Q\left[KI_{\left\{S_1(T)\geq\frac{c(S_2(T))^b}{E_Q\left((S_2(T))^b\right)}\right\}}\right]=K\Phi(d_3). \tag{48}$$

Combining those three term, we get the *BS-approximation* as in equation (11).

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The article was submitted on 17.05.2021, reviewed on 16.07.2021, and accepted for publication on 17.08.2021.
The authors read and approved the final version of the manuscript.
Статья поступила в редакцию 17.05.2021; после рецензирования 16.07.2021; принята к публикации 17.08.2021.
Авторы прочитали и одобрили окончательный вариант рукописи.

ORIGINAL PAPER

DOI:10.26794/2308-944X-2021-9-3-52-76
JEL C02, G13

Proximity of Bachelier and Samuelson Models for Different Metrics

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ABSTRACT

This paper proposes a method of comparing the prices of European options, based on the use of probabilistic metrics, with respect to two models of price dynamics: Bachelier and Samuelson. In contrast to other studies on the subject, we consider two classes of options: European options with a Lipschitz continuous payout function and European options with a bounded payout function. For these classes, the following suitable probability metrics are chosen: the Fortet-Maurier metric, the total variation metric, and the Kolmogorov metric. It is proved that their computation can be reduced to computation of the Lambert in case of the Fortet-Mourier metric, and to the solution of a nonlinear equation in other cases. A statistical estimation of the model parameters in the modern oil market gives the order of magnitude of the error, including the magnitude of sensitivity of the option price, to the change in the volatility.

Keywords: Bachelier model; Samuelson model; option pricing; probabilistic metrics; sensitivity; volatility

For citation: **Smirnov, S., & Sotnikov, D. (2021). Proximity of Bachelier and Samuelson Models for Different Metrics. *Review of Business and Economics Studies, 9*(3),52-76. DOI: 10.26794/2308-944X-2021-9-3-52-76**

Близость моделей Башелье и Самуэльсона для различных метрик

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АННОТАЦИЯ

В статье представлен метод сравнения цен европейских опционов, основанный на использовании вероятностных метрик, применительно к двум моделям динамики цен — Башелье и Самуэльсона. В отличие от других работ на данную тему, рассматриваются классы опционов, а именно европейские опционы с функцией выплат, удовлетворяющих условию Липшица, а также европейские опционы с ограниченной функцией выплат. Для данных классов выбираются подходящие вероятностные метрики: метрика Форте-Мурье, метрика полной вариации и метрика Колмогорова. Мы доказали, что их вычисление сводится к вычислению W-функции Ламберта в случае метрики Форте-Мурье и к решению некоторого нелинейного уравнения в остальных случаях. Статистическая оценка параметров моделей на современном нефтяном рынке указывает на порядок величины погрешности, включая величину чувствительности цены опциона к изменению показателя волатильности.

Ключевые слова: модель Башелье; модель Самуэльсона; ценообразование опционов; вероятностные метрики; чувствительность; волатильность

1 Introduction

Description of Models and Motivation for the Study

In this study, the simplest continuous-time financial market models are considered. The movement of prices $(X_t)_{t\in[0,T]}$ of an asset in the market is described in the framework of the Bachelier model (Bachelier, 1900), using the stochastic Brownian motion process:

$$X_t^B = X_0\left(1+\alpha t+\sigma_B W_t\right), t\in[0,T], \#(1)$$

where $(W_t)_{t\in[0,T]}$ is the Wiener process, $\alpha\in\mathbb{R}, \sigma_B>0$.

The model proposed by Samuelson[1] (1965) uses geometric (economic) Brownian motion to describe the price dynamics:

$$X_t^S = X_0\exp\left[\gamma t+\sigma_S W_t\right], t\in[0,T], \#(2)$$

where $\gamma\in\mathbb{R}, \sigma_S>0$.

In both models, the volatilities σ_B and σ_S are chosen so that they have the dimension $[\text{time}]^{-1/2}$ and the linear trend α and exponential trend γ have the dimension $[\text{time}]^{-1}$.

Hereafter, the prices considered are assumed to be discounted, which is equivalent to a zero risk-free interest rate.

The Black-Scholes (1973) and Merton (1973) option pricing model is based on the Samuelson model (describing price dynamics in the market) and is the most popular in practice. Similarly, for the options on futures Black's (1976) pricing model is based on Samuelson's model.

Bachelier (1900) not only described the dynamics of prices but also built a model of option pricing. However, Samuelson (1965) noted that the stock prices should not be negative; thus, Bachelier's model has not been widely used in practice. Nevertheless, for short-term options, the Bachelier model can better fit the real market data than the Black-Scholes–Samuelson model (e.g., Versluis (2006)). Note that the Bachelier model and its modifications have been applied to modern works on mathematical finance. For example, the Bachelier model and its modification with an absorption screen was used by Glazyrina and Melnikov (2020) for pricing life insurance policies with an invested stock index option, and Melnikov and Wan (2021) compared this model with the Bachelier and Samuelson models.

An unprecedented event occurred on April 20, 2020, when West Texas Intermediate (WTI) crude oil futures prices (the benchmark for U.S. crude oil prices) reached negative levels (see CFTC Interim Staff Report, Trading in NYMEX WTI Crude Oil Futures Contract Leading up to, on, and around April 20, 2020). Fuel supply has far exceeded the demand due to the coronavirus pandemic. Due to overproduction, the storage tanks were so full that it would have been difficult to find room for new oil if the future contracts had been brought to delivery. Because the May contract expired on April 21, market participants with long positions did not want to take delivery of oil (which no one needed at that point in time) and incur storage costs and opted to lock in such large losses by entering into offset deals that the prices turned negative. As of April 22, 2020, the Chicago Mercantile Exchange (CME) switched to the Bachelier pricing model for the options on futures for several energy commodities[2] to account for the possibility of negative prices.

In this regard, it is interesting to compare the prices of derivative financial instruments obtained using the above-described models. Schachermayer and Teichmann (2005) proved the following estimation for the price difference of a call option "at the money" with an expiration at the moment T:

$$0\le C_B-C_S\le\frac{X_0}{12\sqrt{2\pi}}(\sigma\sqrt{T})^3.$$

Here, $\sigma_B=\sigma_S=\sigma$ and C_B, C_S denote the option prices in the Bachelier and Samuelson models, respectively. Both processes (1) and (2) are diffusion processes; thus, the Bachelier and Samuelson models are clearly close in case of small (and equal) values of integral volatility $\sigma_B\sqrt{T}=\sigma_S\sqrt{T}=\sigma\sqrt{T}$. Meanwhile, the Samuelson model is close[3] to the Bachelier model with a linear trend $\gamma+\frac{1}{2}\sigma^2$.

Grunspan (2011) obtained an asymptotic relation between implicit volatilities for normal and lognormal models at $T\to 0$ and compared the sensitivities (greeks) for call options. The differences in option pricing obtained using the Bachelier and Samuelson models are detailed in Thomson (2016).

Another question is for what values of $\sigma_B\sqrt{T}$ and $\sigma_S\sqrt{T}$ models can be considered close? We are interested in the problem of comparing the prices of a European option with an arbitrary payoff function $f(\cdot)$ that belongs to a specific class of functions and depends only on the price X_T of the underlying asset at the time of expiration T. For each of the models (1) and (2), there exists a single equivalent risk-neutral (martingale) measure. The option price $P(f,T)$ with payout function $f(\cdot)$ and time to expiration T is determined as the mathematical expectation relative to the corresponding risk-neutral measure[4]:

$$P(f,T)=\mathbb{E}^* f(X_T).$$

The processes given by relations (1) and (2) are martingales if and only if

$$\alpha=0,\gamma=-\frac{\sigma_S^2}{2}.\#(3)$$

Therefore, the difference between the option prices $P_B(f,T)$ and $P_S(f,T)$ in the Bachelier and Samuelson models can be expressed as follows:

$$P_B(f,T)-P_S(f,T)=Ef\left(X_T^B\right)-Ef\left(X_T^S\right),\#(4)$$

where the process parameters are chosen according to (3).

The estimate for the right part of (4) can be obtained by calculating the distance in the Fortet–Mourier metric between the distributions of random variables X_T^B, X_T^S in case of Lipschitz continuity of the payoff function $f(\cdot)$. If the payout function is discontinuous but bounded (e.g., as in the case of a binary option), the total variation metric can be used for the estimation. However, the Kolmogorov metric can also be used to compare the binary option prices; the closeness of distributions under the total variation metric is a very strong assumption, and hence, the corresponding estimate is rougher (but applicable to a broader class of payout functions).

To compare the Bachelier and Samuelson models, it is interesting to find the optimal relation between the volatilities σ_B, σ_S. Optimality is understood as the dependence between these indicators that arises when minimizing the distance between X_T^B and X_T^S in (one or another) probability metric $d(\cdot,\cdot)$.

In this paper, the Fortet–Mourier metric between random variables X_T^B and X_T^S is calculated and the formulae for the total variation metric and Kolmogorov metric are obtained. The dependence of volatilities that minimizes the Fortet-Mourier metric between X_T^B and X_T^S. Using the probability metrics, the estimates for (4) are obtained to analyze the effect of model choice on option price. By constructing confidence intervals for volatilities in the oil market for standard and binary call and put options, we evaluate the error resulting from the approximate measurement of the volatility.

Notation and Definitions

Let S be a metric space with metric $d(\cdot,\cdot)$ and let us denote by $\mathcal{M}(S)$ the set of all signed measures on S and $\mathcal{P}(S)\subset\mathcal{M}(S)$ as the set of all probability measures on S equipped with Borel σ-algebra.

Definition 1. *Let us define a semi-norm in the space $Lip(S)$ of the Lipschitz continuous on S functions as follows:*

$$\| f \|_{Lip}=\sup_{x,y}\frac{|f(x)-f(y)|}{d(x,y)}, f(\cdot)\in Lip(S).$$

Definition 2. *In the space $B(S)$ of bounded measurable functions on S, let us define the norm*

$$\| f \|_B=\sup_{x\in S}|f(x)|, f(\cdot)\in B(S).$$

Definition 3. *For $S=\mathbb{R}$ in the space $St(\mathbb{R})$ of piecewise constant functions with finite number of jumps $\Delta_1,\ldots,\Delta_m$, we define a semi-norm*

$$\| f \|_{St}=\sum_{j=1}^{m}|\Delta_j|, f(\cdot)\in St(\mathbb{R}).$$

The introduced semi-norm is a norm in space $St(\mathbb{R})/\mathbb{R}$.

Definition 4. *By the coupling of two random variables X u Y, we call[5] a pair (X',Y') for which the following is true $X'\overset{d}{=}X, Y'\overset{d}{=}Y$. For the monotone coupling of real random variables X u Y with distribution functions $F_X(\cdot), F_Y(\cdot)$, we call a pair of*

$$\left(F_X^{-1}(U),F_Y^{-1}(U)\right),U\sim\mathcal{U}(0,1),$$

where F_X is the distribution function of a random variable X, which is defined as

$$F_X(x)=\mathbb{P}(X<x),$$

and F^{-1} *is the generalized inverse function of the monotonically non-decreasing left-continuous function, defined via the relation*[6]

$$F^{-1}(y) = \inf\{x \in \mathbb{R} : F(x) \geq y\} = \\ = \sup\{x \in \mathbb{R} : F(x) < y\}, y \in (0,1).$$

Let $\delta(\cdot,\cdot)$ be a metric in the space of random variables taking values in S, defined on pairs of (X,Y) of random variables, with a common probability space.

Definition 5. *The minimal metric with respect to* $\delta(\cdot,\cdot)$ *is the metric*

$$\hat{\delta}(X,Y) = \inf\left\{\delta(X',Y') : X' \stackrel{d}{=} X, Y' \stackrel{d}{=} Y\right\}.$$

Note that $\hat{\delta}(\cdot,\cdot)$ is therefore a metric in the space of distributions and does not depend on the joint distribution of X and Y.

Let $\mathcal{F}$ be a set of measurable functions $f : S \to \mathbb{R}$. Then, for each signed measure μ on S such that $\int_S |f||d\mu| < \infty$ for all $f \in \mathcal{F}$, the following semi-norm can be defined:

$$\|\mu\|_{\mathcal{F}}^{*} = \sup_{f \in \mathcal{F}} \left|\int_S f d\mu\right|.$$

Denote $\mathcal{M}_{\mathcal{F}} = \{\mu \in \mathcal{M}(S) : \|\mu\|_{\mathcal{F}}^{*} < \infty\}$.

Definition 6. *We can say that on* $\mathcal{M}_{\mathcal{F}}$ *the dual semimetric if*

$$d_{\mathcal{F}}(\mu,\nu) = \|\mu - \nu\|_{\mathcal{F}}^{*}.$$

In particular, for the probabilistic measures $\mathcal{P}_{\mathcal{F}} = \mathcal{M}_{\mathcal{F}} \cap \mathcal{P}(S)$,

$$d_{\mathcal{F}}(X,Y) = \sup_{f \in \mathcal{F}} \left|\mathbb{E}f(X) - \mathbb{E}f(Y)\right|.$$

Let $(S,\mathcal{B})$ be a measurable space.

Definition 7. *The total variation norm for a signed measure* μ *is defined as*

$$\|\mu\|_{TV} = \sup\left\{\int_S f d\mu : f \in B(S), \|f\|_B \leq 1\right\}.$$

Definition 8. *A total variation metric is a probability metric*

$$d_{TV}(Q_1,Q_2) = \|Q_1 - Q_2\|_{TV}.$$

If distributions Q_1, Q_2 are absolutely continuous with respect to the measure μ and have Radon–Nikodym densities $p_1(\cdot), p_2(\cdot)$, then

$$d_{TV}(Q_1,Q_2) = \int_S |p_1(x) - p_2(x)| \mu(dx) = \\ = 2\int_S (p_1(x) - p_2(x))^{+} \mu(dx), \# (5)$$

where $a^{+} = \max(a,0)$.

Definition 9. *If* $S = \mathbb{R}$*, then the Kolmogorov metric*[7] *is*

$$d_K(X,Y) = \sup_{x \in \mathbb{R}} |F_X(x) - F_Y(x)|.$$

Definition 10. *The Fortet-Mourier metric*[8] *is the probability metric*

$$d_{FM}(X,Y) = \sup_{\|f\|_{Lip} \leq 1} \left|\mathbb{E}f(X) - \mathbb{E}f(Y)\right|.$$

There is also an equivalent representation of this metric:

$$d_{FM}(X,Y) = \min\left\{\mathbb{E}d(X',Y') : X' \stackrel{d}{=} X, Y' \stackrel{d}{=} Y\right\}. \# (6)$$

The proof of equivalence of the definitions can be found in Rachev, Klebanov, Stoyanov, and Fabozzi (2013).

It has been shown (e.g., Bogachev (2007)) that in case of $S = \mathbb{R}$, the minimum value in (6) is attained on the monotone coupling

$$\left(F_X^{-1}(U), F_Y^{-1}(U)\right), U \sim \mathcal{U}(0,1).$$

Remark 1. *The Fortet-Mourier metric allows one to derive an upper estimate of (4) in the case of Lipschitz continuity of* $f(\cdot)$*, for example, if* $f(\cdot)$ *is piecewise linear (which corresponds to the portfolio of call and put options). It is also possible to estimate (4) by using the total variation metric if the function* $f(\cdot)$ *is bounded. Even if the payout function is neither Lipschitz continuous nor bounded (e.g., if it corresponds to a portfolio of binary and call options), it can most likely be represented as a sum of ones, as in practice, the payout functions usually do not grow faster than linear ones. The Kolmogorov metric provides a more accurate estimate than the total variation metric; however, it is only applicable to piecewise constant payout functions corresponding to a portfolio composed of binary options.*

Definition 11. Lambert W *function is a complex-valued function* $W : \mathbb{C} \to \mathbb{C}$*, defined as a solution of the equation* $z = W(z)e^{W(z)}, z \in \mathbb{C}$.

$W(\cdot)$ cannot be expressed in elementary functions. We are only interested in its two branches, $W_0(z), W_{-1}(z)$, at $z \in (-e^{-1}, 0)$ (Fig. 1), which correspond to the real solutions of the equation

$$xe^x = z, z \in (-e^{-1}, 0).$$

The definition and notation are taken from Corless, Gonnet, Hare, Jeffrey, and Knuth (1996).

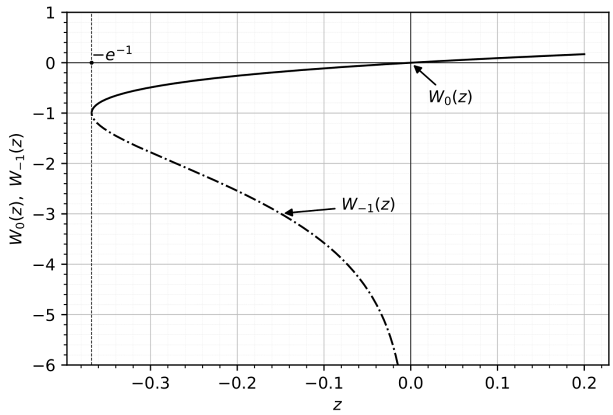

Figure 1. Real-valued branches of Lambert W -function

Source: The authors.

2 Main Results

Let us show how one can obtain the estimates for (4) by using the introduced probability metrics. Let, as mentioned above, $P_B(f,T), P_S(f,T)$ stand for the prices of European options with payoff function $f(\cdot)$ and time to expiration T in the Bachelier and Samuelson models, respectively. Then, the following estimates are true:

If $f(\cdot) \in Lip(\mathbb{R})$, then

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f \|_{Lip}\, d_{FM}\left(X_T^B, X_T^S\right). \# (7)$$

If $f(\cdot) \in B(\mathbb{R})$, then

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f \|_{B}\, d_{TV}\left(X_T^B, X_T^S\right). \# (8)$$

If $f(\cdot) \in St(\mathbb{R})$, then

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f \|_{St}\, d_{K}\left(X_T^B, X_T^S\right). \# (9)$$

Indeed, the price of a European option is defined in the Bachelier and Samuelson models as a mathematical expectation of the payout function relative to the risk-neutral measure:

$$P_B(f,T) = \mathbb{E}f\left(X_T^B\right), P_S(f,T) = \mathbb{E}f\left(X_T^S\right),$$

where the processes X_t^B, X_t^S are martingales, i.e., $\alpha = 0, \gamma = -\frac{\sigma_S^2}{2}$.

Then,

$$\left|P_B(f,T) - P_S(f,T)\right| = \left|\mathbb{E}\left(f\left(X_T^B\right) - f\left(X_T^S\right)\right)\right|.$$

1. In case of Lipschitz continuity of $f(\cdot)$,

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f \|_{Lip} \sup_{\|g\|_{Lip} \le 1} \left|\mathbb{E}g\left(X_T^B\right) - \mathbb{E}g\left(X_T^S\right)\right| = \| f \|_{Lip}\, d_{FM}\left(X_T^B, X_T^S\right)$$

2. If $f(\cdot)$ is bounded, then

$$\left|P_B(f,T) - P_S(f,T)\right| = \left|\int_{\mathbb{R}} f(x)\left(p_{X_T^B}(x) - p_{X_T^S}(x)\right)dx\right| \le$$

$$\le \| f \|_B \int_{\mathbb{R}} \left|p_{X_T^B}(x) - p_{X_T^S}(x)\right| dx = \| f \|_B\, d_{TV}\left(X_T^B, X_T^S\right).$$

Here, $p_{X_T^B}(\cdot), p_{X_T^S}(\cdot)$ denote the densities of random variables X_T^B, X_T^S.

3. The function $f(\cdot) \in St(\mathbb{R})$ can be represented as

$$f(X_T) = f(-\infty) + \sum_{j=1}^{m} f_j(X_T), f_j(x) = \Delta_j \mathbb{I}_{x>K_j}.$$

For each function, $f_j(\cdot)$ it is true that

$$\left|P_B(f_j,T) - P_S(f_j,T)\right| = \left|\Delta_j\right| \left|F_{X_T^S}(K_j) - F_{X_T^B}(K_j)\right| \le \left|\Delta_j\right| d_K\left(X_T^B, X_T^S\right).$$

$$\left|P_B(f,T) - P_S(f,T)\right| \le \sum_{j=1}^{m} \left| P_B(f_j,T) - P_S(f_j,T)\right| \le \sum_{j=1}^{m} \left|\Delta_j\right| d_K\left(X_T^B, X_T^S\right) = \\ = \| f \|_{St} \, d_K\left(X_T^B, X_T^S\right).$$

Note 2: *If the payout function can be represented as*

$$f(\cdot) = f_1(\cdot) + f_2(\cdot) + f_3(\cdot), f_1(\cdot) \in Lip(\mathbb{R}), f_2(\cdot) \in B(\mathbb{R}), f_3(\cdot) \in St(\mathbb{R}), \#(10)$$

then

$$\left|P_B(f,T) - P_S(f,T)\right| \le \| f_1 \|_{Lip} \, d_{FM}\left(X_T^B, X_T^S\right) + \| f_2 \|_B \, d_{TV}\left(X_T^B, X_T^S\right) + \| f_3 \|_{St} \, d_K\left(X_T^B, X_T^S\right). \#(11)$$

The representation (10) is obviously not unique. Moreover, $f_3(\cdot)$ is unnecessary as soon as any piecewise constant function with a finite number of jumps is bounded. Nevertheless, a proper choice of functions $f_1(\cdot), f_2(\cdot)$ и $f_3(\cdot)$ in expansion (10) can significantly improve the estimate (11).

The following statements provide methods of calculation of the metrics appearing in (7)–(9).

Finding $d_{FM}\left(X_t^B, X_t^S\right)$ is reduced to the calculation of the metric between random variables $\xi \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$ and $\eta \sim \mathcal{LN}\left(\mu_2, \sigma_2^2\right)$ that have normal and lognormal distributions. The value of this metric is given by the following theorem.

Theorem 1
Let $\xi \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right), \eta \sim \mathcal{LN}\left(\mu_2, \sigma_2^2\right)$. Then, under the condition $\ln\left(\frac{\sigma_2}{\sigma_1}\right) + \mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1 + 1 < 0, \#(*)$,

the metric can be found with the formula

$$d_{FM}(\xi,\eta) = \mu_1\left(2\left[\Phi(k_2) - \Phi(k_1)\right] - 1\right) + 2\sigma_1\left(\phi(k_1) - \phi(k_2)\right) + \\ + \exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right]\left(1 - 2\left[\Phi(k_2 - \sigma_2) - \Phi(k_1 - \sigma_2)\right]\right), \#(12)$$

where $\Phi(\cdot)$ is a cumulative distribution function of the standard normal distribution, $\phi(\cdot)$ is the density of the standard normal distribution, and k_1 and k_2 are equal to

$$k_1 = -\frac{\mu_1}{\sigma_1} - \frac{1}{\sigma_2} W_0\left(-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1\right]\right), \\ k_2 = -\frac{\mu_1}{\sigma_1} - \frac{1}{\sigma_2} W_{-1}\left(-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1\right]\right). \#(13)$$

If condition (*) is not satisfied, then

$$d_{FM}(\xi,\eta) = -\mu_1 + \exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right]. \#(14)$$

Corollary 1. *When trends and volatilities are chosen such that processes (1) and (2) are martingales (i.e., relation (3) is satisfied), the formula for the metric between distributions of the random variables* X_t^B, X_t^S *can be expressed as*

$$d_{FM}\left(X_t^B, X_t^S\right) = 2X_0(\left[\Phi(k_2)-\Phi(k_1)\right]-\left[\Phi(k_2-\sigma_2)-\Phi(k_1-\sigma_2)\right] \\ -\left[\phi(k_2)-\phi(k_1)\right]), \#(15)$$

where $\sigma_1 = \sigma_B\sqrt{t}, \sigma_2 = \sigma_S\sqrt{t}$ *denote the integral volatilities, and* k_1, k_2 *are calculated as follows:*

$$\begin{aligned} k_1 &= -\frac{1}{\sigma_1}-\frac{1}{\sigma_2}W_0\left(-\frac{\sigma_2}{\sigma_1}\exp\left[-\frac{\sigma_2^2}{2}-\frac{\sigma_2}{\sigma_1}\right]\right), \\ k_2 &= -\frac{1}{\sigma_1}-\frac{1}{\sigma_2}W_{-1}\left(-\frac{\sigma_2}{\sigma_1}\exp\left[-\frac{\sigma_2^2}{2}-\frac{\sigma_2}{\sigma_1}\right]\right). \end{aligned} \#(16)$$

The following theorem answers the question about the optimal relation between σ_B and σ_S minimize the Fortet–Mourier metric in the risk-neutral case.

Theorem 2
For fixed σ_2, *the minimum of expression (15) is attained at*

$$\sigma_1^* = \frac{\sigma_2\sqrt{1-e^{-\sigma_2^2}}}{\frac{\sigma_2^2}{2}+\ln\left(1+\sqrt{1-e^{-\sigma_2^2}}\right)}.$$

For fixed σ_1, *the minimum in (15) is attained at* σ_2^*, *which is a solution of the equation* $k_1+k_2=2\sigma_2$, *where* k_1, k_2 *are determined from (16).*

The calculation of the total variation metric and the Kolmogorov metric between X_T^B and X_T^S can be reduced to solving a nonlinear equation. This result is formulated in Theorem 3.

Theorem 3
L $\xi \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right), \eta \sim \mathcal{LN}\left(\mu_2, \sigma_2^2\right)$, *and* $\mu_1 = 1, \mu_2 = -\frac{\sigma_2^2}{2}$. *Then,*

$$d_{TV}(\xi,\eta) = 2\left[\left(F_\xi(x_1)-F_\eta(x_1)\right)+\left(F_\xi(x_3)-F_\eta(x_3)\right)-\left(F_\xi(x_2)-F_\eta(x_2)\right)\right], \#(17)$$

$$d_K(\xi,\eta) = \max_{i=1,2,3}\left|F_\xi(x_i)-F_\eta(x_i)\right|, \#(18)$$

where $x_1 \le x_2 \le x_3$ *are the roots of the equation*

$$\left(x^2-2x\right)-\left(\frac{\sigma_1}{\sigma_2}\right)^2(\ln x)^2-3\sigma_1^2\ln x+1-\frac{\sigma_1^2\sigma_2^2}{4}-2\sigma_1^2\ln\left(\frac{\sigma_2}{\sigma_1}\right)=0, \#(19)$$

and the cumulative distribution functions have the form $F_\xi(x)=\Phi\left(\frac{x-\mu_1}{\sigma_1}\right), F_\eta(x)=\Phi\left(\frac{\ln(x)-\mu_2}{\sigma_2}\right)\mathbb{I}_{x\ge 0}$.

Corollary 2. *According to Definitions* 8 *and* 9,

$$d_{TV}\left(X_T^B, X_T^S\right) = d_{TV}\left(\frac{X_T^B}{X_0}, \frac{X_T^S}{X_0}\right), d_K\left(X_T^B, X_T^S\right) = d_K\left(\frac{X_T^B}{X_0}, \frac{X_T^S}{X_0}\right).$$

In the risk-neutral case, $\frac{X_T^B}{X_0} \sim \mathcal{N}\left(1, \sigma_S^2 T\right), \frac{X_T^S}{X_0} \sim \mathcal{LN}\left(-\frac{\sigma_S^2 T}{2}, \sigma_S^2 T\right)$, *and the metrics are calculated by Theorem* 3 *by taking into account that* $\sigma_1 = \sigma_B \sqrt{T}, \sigma_2 = \sigma_S \sqrt{T}$.

3 Proofs of Theorems

Proof of Theorem 1

The cumulative distribution functions of ξ, η are

$$F_\xi(x) = \Phi\left(\frac{x-\mu_1}{\sigma_1}\right), F_\eta(x) = \Phi\left(\frac{\ln(x)-\mu_2}{\sigma_2}\right)\mathbb{I}_{x \geq 0}.$$

Then, their inverse functions can be expressed as

$$F_\xi^{-1}(u) = \mu_1 + \sigma_1 \Phi^{-1}(u), F_\eta^{-1}(u) = e^{\mu_2 + \sigma_2 \Phi^{-1}(u)}.$$

As the minimum in (6) is attained on the monotone coupling, we obtain

$$d_{FM}(\xi,\eta) = \mathbb{E}\left|(\mu_1 + \sigma_1 Z) - e^{\mu_2 + \sigma_2 Z}\right|, Z = \Phi^{-1}(U) \sim \mathcal{N}(0,1).$$

The expectation is considered here with respect to the measure $\mathbb{P}_Z$ induced by a random variable Z.
Let us divide the space of elementary events into three disjoint sets:

$$D_1 = \left\{\omega : \mu_1 + \sigma_1 Z > e^{\mu_2 + \sigma_2 Z}\right\},$$

$$D_2 = \left\{\omega : \mu_1 + \sigma_1 Z < e^{\mu_2 + \sigma_2 Z}\right\},$$

$$D_3 = \left\{\omega : \mu_1 + \sigma_1 Z = e^{\mu_2 + \sigma_2 Z}\right\}.$$

As $\mathbb{P}(D_3) = 0,\ \mathbb{P}(D_1 \bigcup D_2) = 1 holds,$ and therefore,

$$d_{FM}(\xi,\eta) = \mathbb{E}\left[(\mu_1 + \sigma_1 Z) - e^{\mu_2 + \sigma_2 Z}\right]\mathbb{I}_{D_1} + \mathbb{E}\left[e^{\mu_2 + \sigma_2 Z} - (\mu_1 + \sigma_1 Z)\right]\mathbb{I}_{D_2}.$$

By definition, the set D_1 is either empty or comprises those ω for which $Z \in (k_1, k_2)$ for some real k_1, k_2 as the graph of a linear function can lie above the graph of an exponent only within a finite interval.

In case of $D_1 = \varnothing$, considering that the expectation of the lognormal distribution with parameters μ_2, σ_2^2 is equal to $\exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right]$, we obtain

$$d_{FM}(\xi,\eta) = E\left[e^{\mu_2 + \sigma_2 Z} - (\mu_1 + \sigma_1 Z)\right] = -\mu_1 + \exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right]. \#(20)$$

If $D_1 = \left\{\omega : Z \in (k_1, k_2)\right\}$, then as it is much more convenient to work with D_1 than with D_2, we eliminate the indicator $\mathbb{I}_{D_2}$. Using the formula

$$\mathbb{E}X\mathbb{I}_{D_2} = \mathbb{E}X - \mathbb{E}X\mathbb{I}_{D_1}$$

for $X = e^{\mu_2 + \sigma_2 Z} - (\mu_1 + \sigma_1 Z)$, we get

$$d_{FM}(\xi,\eta) = -\mu_1 + \exp\left[\mu_2 + \frac{\sigma_2^2}{2}\right] - 2E\left[e^{\mu_2 + \sigma_2 Z} - (\mu_1 + \sigma_1 Z)\right]\mathbb{I}_{D_1}. \#(21)$$

As $\mathbb{P}(D_1) = \Phi(k_2) - \Phi(k_1)$, we need to calculate $\mathbb{E}Z\mathbb{I}_{D_1}$ и $\mathbb{E}e^{\sigma_2 Z}\mathbb{I}_{D_1}$.

To find the first moment of a random variable $Z\mathbb{I}_{D_1}$, we find its Laplace transform

$$\psi(\lambda) = Ee^{-\lambda Z\mathbb{I}_{D_1}} = 1 - P(D_1) + \frac{1}{\sqrt{2\pi}}\int_{k_1}^{k_2}\exp\left[-\lambda x - \frac{x^2}{2}\right]dx =$$

$$= 1 - P(D_1) + \exp\left[\frac{\lambda^2}{2}\right]\left[\Phi(k_2 + \lambda) - \Phi(k_1 + \lambda)\right].$$

As the first moment exists, it is equal to

$$\mathbb{E}Z\mathbb{I}_{D_1} = -\psi'(0) = \phi(k_1) - \phi(k_2).$$

Now, let us find

$$\mathbb{E}e^{\sigma_2 Z}\mathbb{I}_{D_1} = \frac{1}{\sqrt{2\pi}}\int_{k_1}^{k_2}\exp\left[\sigma_2 x - \frac{x^2}{2}\right]dx = \exp\left[\frac{\sigma_2^2}{2}\right]\left[\Phi(k_2 - \sigma_2) - \Phi(k_1 - \sigma_2)\right].$$

Combining the above formulas, we obtain

$$d_{FM}(\xi,\eta) = \mu_1\left(2\left[\Phi(k_2) - \Phi(k_1)\right] - 1\right) + 2\sigma_1\left[\phi(k_1) - \phi(k_2)\right] +$$

$$+ e^{\mu_2 + \frac{\sigma_2^2}{2}}\left(1 - 2\left[\Phi(k_2 - \sigma_2) - \Phi(k_1 - \sigma_2)\right]\right).$$

To obtain the final result, it is necessary to calculate k_1, k_2 and find the conditions under which D_1 is nonempty. If D_1 is nonempty, then k_1, k_2 are the roots of the equation

$$\mu_1 + \sigma_1 x = \exp\left[\mu_2 + \sigma_2 x\right]. \# (22)$$

Now, let us make the variable replacement $y = -\frac{\sigma_2}{\sigma_1}(\mu_1 + \sigma_1 x), x = -\frac{\mu_1}{\sigma_1} - \frac{y}{\sigma_2}$. Then, the equation is transformed into

$$-\frac{\sigma_1}{\sigma_2}y = \exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1} - y\right];$$

$$ye^y = -\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1}\right]. \# (23)$$

The right-hand side is negative, so (23) has two real solutions (i.e., D_1 is nonempty) only in the case of $-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1}\right] > -e^{-1}$ (see the definition of the Lambert W). Taking the logarithm of this inequality, we obtain (*).

If condition (*) is satisfied, the roots of (23) are found using the W function:

$$y_1 = W_0\left(-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1}\right]\right),$$

$$y_2 = W_{-1}\left(-\frac{\sigma_2}{\sigma_1}\exp\left[\mu_2 - \mu_1\frac{\sigma_2}{\sigma_1}\right]\right).$$

By substituting these solutions into the inverse replacement $x=-\frac{\mu_1}{\sigma_1}-\frac{y}{\sigma_2}$, we obtain (13), which completes the proof of the theorem.

Proof of Corollary 1

If X, Y are random variables, it immediately follows from (6) that

$$d_{FM}(cX, cY)=|c| d_{FM}(X, Y), c\in\mathbb{R}.$$

Thus,
$$d_{FM}\left(X_t^B, X_t^S\right)=X_0 d_{FM}\left(1+\sigma_B W_t, \exp\left[-\frac{\sigma_S^2 t}{2}+\sigma_S W_t\right]\right)=X_0 d_{FM}(\xi,\eta).$$

Here, we designate $\xi=1+\sigma_B W_t, \eta=\exp\left[-\frac{\sigma_S^2 t}{2}+\sigma_S W_t\right]$. Clearly,

$$\xi\sim N\left(\mu_1,\sigma_1^2\right), \eta\sim LN\left(\mu_2,\sigma_2^2\right), \#(24)$$

where $\mu_1=1, \mu_2=-\frac{\sigma_S^2 t}{2}, \sigma_1^2=\sigma_B^2 t, \sigma_2^2=\sigma_S^2 t$.

Let us show that condition (*) is satisfied. Suppose that for some $\sigma_1>0, \sigma_2>0$, this is not true. Then, through (14), $d_{FM}(\xi,\eta)=0$ (i.e., $\xi \overset{d}{=} \eta$). We obtain the contradiction with (24). Substituting the parameter values into formula (12) of Theorem 1, we obtain (15) and (16).

Proof of Theorem 2

1. Let us fix $\sigma_2>0$ and consider an optimization problem

$$d_{FM}(\xi,\eta)\to\min_{\sigma_1>0}$$

From (15) and (16) and the continuous differentiability of W for $\sigma_1,\sigma_2>0$, the function $d_{FM}(\xi,\eta)$ is found to be continuously differentiable with respect to σ_1 at $\sigma_1,\sigma_2>0$. Moreover, the values close to zero and a very large value of σ_1 are not optimal. Hence, the minimum point satisfies the necessary condition

$$\frac{\partial d_{FM}(\xi,\eta)}{\partial\sigma_1}=0.$$

Substituting into (21) the martingale values of parameters and differentiating it by σ_1 using the Leibniz integral rule, we obtain

$$\frac{\partial d_{FM}(\xi,\eta)}{\partial\sigma_1}=-2\frac{\partial}{\partial\sigma_1}\mathbb{E}\left[\exp\left[-\frac{\sigma_2^2}{2}+\sigma_2 Z\right]-1-\sigma_1 Z\right]\mathbb{I}_{D_1}=$$
$$=-2\frac{\partial}{\partial\sigma_1}\int_{k_1}^{k_2}\left(\exp\left[-\frac{\sigma_2^2}{2}+\sigma_2 z\right]-1-\sigma_1 z\right)\phi(z)dz=2\int_{k_1}^{k_2} z\phi(z)dz-$$
$$-2\phi(z)\left(\exp\left[-\frac{\sigma_2^2}{2}+\sigma_2 z\right]-1-\sigma_1 z\right)\Big|_{k_1}^{k_2}=2\mathbb{E}Z\mathbb{I}_{D_1}=2\left[\phi(k_1)-\phi(k_2)\right].$$

Here, the term with substitution is equal to zero, as k_1, k_2 are the roots of (22).

Thus, the point σ_1 is optimal if and only if

$$\phi(k_1)=\phi(k_2)\Leftrightarrow|k_1|=|k_2|.$$

Let us show that the case $k_1 = k_2$ is impossible. Indeed, if $k_1 = k_2$, then from (15), $d_{FM}(\xi,\eta) = 0$; that is, $\xi \overset{d}{=} \eta$. We obtain the contradiction with

$$\xi \sim \mathcal{N}(\mu_1, \sigma_1^2), \eta \sim \mathcal{LN}(\mu_2, \sigma_2^2).$$

Thus, $k_2 = -k_1$. From (16), we obtain

$$W_0(z) + W_{-1}(z) = -2\delta,$$

where we designate $\delta = \frac{\sigma_2}{\sigma_1}, z = -\frac{\sigma_2}{\sigma_1}\exp\left[-\frac{\sigma_2^2}{2} - \frac{\sigma_2}{\sigma_1}\right]$. Adding to this equation the definition of the Lambert W function, we obtain the system

$$\begin{cases} W_0(z) + W_{-1}(z) = -2\delta \\ W_0(z)e^{W_0(z)} = z \\ W_{-1}(z)e^{W_{-1}(z)} = z \end{cases}$$

Solving it, we determine

$$W_0(z) = -\delta + \sqrt{\delta^2 - (ze^{\delta})^2},$$

$$W_{-1}(z) = -\delta - \sqrt{\delta^2 - (ze^{\delta})^2}.$$

Hence, from (16)

$$k_2 = -k_1 = \frac{1}{\sigma_1}\sqrt{1 - e^{-\sigma_2^2}}.$$

Let us substitute the determined value of k_2 into (22)

$$1 + \sqrt{1 - e^{-\sigma_2^2}} = \exp\left[-\frac{\sigma_2^2}{2} + \frac{\sigma_2}{\sigma_1}\sqrt{1 - e^{-\sigma_2^2}}\right].$$

From this, we can easily express as

$$\sigma_1^* = \frac{\sigma_2\sqrt{1 - e^{-\sigma_2^2}}}{\frac{\sigma_2^2}{2} + \ln\left(1 + \sqrt{1 - e^{-\sigma_2^2}}\right)}.$$

2. Analogically to the first point, we equate to zero the derivative

$$\frac{\partial d_{FM}(\xi,\eta)}{\partial \sigma_2} = -2\frac{\partial}{\partial \sigma_1}\mathbb{E}\left[\exp\left[-\frac{\sigma_2^2}{2} + \sigma_2 Z\right] - 1 - \sigma_1 Z\right]\mathbb{I}_{D_1} =$$

$$= -2\int_{k_1}^{k_2}(z - \sigma_2)\exp\left[-\frac{\sigma_2^2}{2} + \sigma_2 z\right]\phi(z)dz = -2\int_{k_1}^{k_2}(z - \sigma_2)\phi(z - \sigma_2)dz =$$

$$= -2\int_{k_1-\sigma_2}^{k_2-\sigma_2} y d\Phi(y) = -2\left[\phi(k_1 - \sigma_2) - \phi(k_2 - \sigma_2)\right] = 0.$$

From here, $|k_1-\sigma_2|=|k_2-\sigma_2|$. Again, considering the impossibility of case $k_1=k_2$, we obtain $k_1+k_2=2\sigma_2$.

Proof of Theorem 3

From (5), we obtain

$$d_{TV}(\xi,\eta)=2\int_A\left(p_\xi(x)-p_\eta(x)\right)dx, \#(25)$$

where set $A=\{x: p_\xi(x)>p_\eta(x)\}$ — is the union of intervals whose endpoints are the roots of the equation

$$p_\xi(x)=p_\eta(x).$$

This equation has only positive roots as $p_\xi(x)>0, p_\eta(x)=0$ at $x\le 0$. Let us write it out explicitly and transform it.

$$\frac{1}{\sigma_1}\exp\left[-\frac{(x-1)^2}{2\sigma_1^2}\right]=\frac{1}{\sigma_2 x}\exp\left[-\frac{(\ln x+\sigma_2^2/2)^2}{2\sigma_2^2}\right];$$

$$\ln x+\ln\left(\frac{\sigma_2}{\sigma_1}\right)=\frac{1}{2\sigma_1^2}\left(x^2-2x+1\right)-\frac{1}{2\sigma_2^2}\left((\ln x)^2+\sigma_2^2\ln x+\frac{\sigma_2^4}{4}\right);$$

$$2\sigma_1^2\ln x+2\sigma_1^2\ln\left(\frac{\sigma_2}{\sigma_1}\right)=\left(x^2-2x\right)+1-\frac{\sigma_1^2}{\sigma_2^2}(\ln x)^2-\sigma_1^2\ln x-\frac{\sigma_1^2\sigma_2^2}{4};$$

$$\left(x^2-2x\right)-\left(\frac{\sigma_1}{\sigma_2}\right)^2(\ln x)^2-3\sigma_1^2\ln x+1-\frac{\sigma_1^2\sigma_2^2}{4}-2\sigma_1^2\ln\left(\frac{\sigma_2}{\sigma_1}\right)=0.$$

Let us denote the left part of the equation by $h(x)$ and find the derivatives of this function:

$$h'(x)=2(x-1)-\left(\frac{\sigma_1}{\sigma_2}\right)^2\frac{2\ln x}{x}-\frac{3\sigma_1^2}{x},$$
$$h''(x)=2-\left(\frac{\sigma_1}{\sigma_2}\right)^2\frac{2(1-\ln x)}{x^2}+\frac{3\sigma_1^2}{x^2}.$$

Equality $h'(x)=0$ is equivalent to $2x(x-1)=2\left(\frac{\sigma_1}{\sigma_2}\right)^2\ln x+3\sigma_1^2$, which has exactly two roots for geometric reasons. Hence, the function $h(x)$ has two local extrema on $(0,+\infty)$. Let us denote them by x_1^*, x_2^* and $x_1^*<x_2^*$.

As $\lim\limits_{x\to 0+}h(x)=-\infty, \lim\limits_{x\to+\infty}h(x)=+\infty$, the equation $h(x)=0$ has $(0,+\infty)$ at most three roots. As $p_\xi(x)>p_\eta(x)$ at $x<0$ and at $x>0$ $p_\xi(x)>p_\eta(x)$, when $h(x)<0$, set A can be represented as

$$A=(-\infty,x_1)\cup(x_2,x_3). \#(26)$$

If the equation has less than three roots, consider $x_2=x_3$. Combining (26) with the integral representation of the total variation metric (25), we obtain the required statement.

To find the Kolmogorov metric, consider the function $g(x)=F_\xi(x)-F_\eta(x)$. As $\lim\limits_{x\to\infty}g(x)=0$ at the point at which the maximum of the modulus is reached, we have the equality $g'(x)=p_\xi(x)-p_\eta(x)=0$.

The solutions of this equation are the roots of x_1,x_2,x_3 obtained in (19). Hence, $d_K(\xi,\eta)=\max\limits_{x\in\mathbb{R}}\left|F_\xi(x)-F_\eta(x)\right|=\max\limits_{i=1,2,3}\left|F_\xi(x_i)-F_\eta(x_i)\right|.$

4 Numerical Analysis

Calculation of the Fortet-Mourier Metric

The value of the Fortet-Mourier metric in (12) cannot be expressed in elementary functions. This is an expected result, which naturally arises when dealing with normal and lognormal distributions: the distribution function $\Phi(\cdot)$ appears, for example, in the Black-Scholes formula (Black and Scholes, 1973). However, in (12) the Lambert W, which is much less frequently used function than $\Phi(\cdot)$. Nevertheless, many mathematical packages allow calculating the value of any of its branches, which simplifies the numerical calculation of the formula.

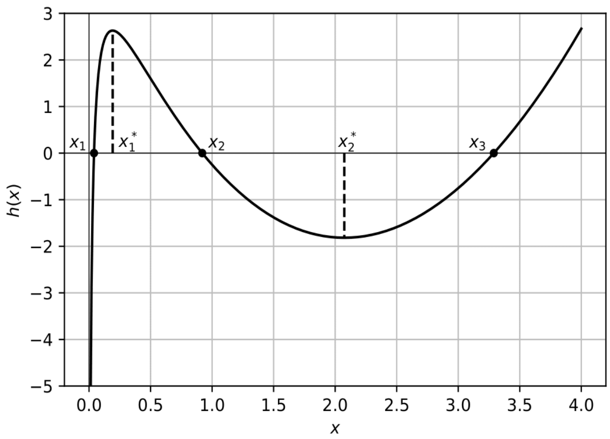

Figure 2. Function graph $h(\cdot)$ at $\sigma_1 = \sigma_2 = 1$

Source: The authors.

Calculation of the Total Variation Metric and the Kolmogorov Metric

Let us discuss here the numerical computation of the total variation metric.

Calculation $d_{TV}(\xi,\eta)$ Using Quadrature Methods

One of the approaches for the calculation of the total variation metric is the calculation (see (5)) of the integral

$$2\int_{\mathbb{R}} (p_\xi(x) - p_\eta(x))^+ \, dx$$

using quadrature methods.

As

$$(p_\xi(x) - p_\eta(x))^+ \le p_\xi(x),$$

and $\xi \sim \mathcal{N}(\mu_1, \sigma_1)$, we will approximate the integral by the proper one

$$\int_{\mathbb{R}} (p_\xi(x) - p_\eta(x))^+ \, dx \approx \int_{1-\delta}^{1+\delta} (p_\xi(x) - p_\eta(x))^+ \, dx.$$

As for $x < 0$,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{t^2}{2}\right] dt \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \frac{t}{x} \exp\left[-\frac{t^2}{2}\right] dt = -\frac{1}{\sqrt{2\pi} x} \exp\left[-\frac{x^2}{2}\right],$$

the approximation error does not exceed

$$2\int_{-\infty}^{1-\delta} p_\xi(x) dx = 2\Phi\left(-\frac{\delta}{\sigma_1}\right) \le \frac{\sigma_1}{\delta} \sqrt{\frac{2}{\pi}} \exp\left[-\frac{\delta^2}{2\sigma_1^2}\right]. \#(27)$$

Now, let us estimate the accuracy of the integral calculation

$$\int_{1-\delta}^{1+\delta} (p_\xi(x) - p_\eta(x))^+ \, dx$$

by using the trapezoidal method. The integrand function here is not twice continuously differentiable; however, (26) indicates that it has no more than three break points. As the function is zero at each break point, the integration error in the mesh section containing these points does not exceed $3M_1h^2$, where

$$M_1 = \max_{x\in(1-\delta,1+\delta)}\left|p_{\xi}^{'}(x)-p_{\eta}^{'}(x)\right|.$$

Combining this with the standard estimation for the trapezoidal rule (Samarsky and Gulin, 1989), we obtain

$$|\Psi| \le \frac{h^2(2\delta)}{12}M_2 + 3M_1h^2,$$

where Ψ is the error incurred in the integration calculations performed on a grid of size N, $h=\frac{2\delta}{N}$ grid step, and $M_2 = \max_{x\in(1-\delta,1+\delta)}\left|p''_{\xi}(x)-p''_{\eta}(x)\right|$.

As $p_{\xi}(x)=\frac{1}{\sigma_1}\phi\left(\frac{x-1}{\sigma_1}\right)$ considering $\phi'(x)=-x\phi(x)$, we find

$$p_{\xi}^{'}(x)=-\frac{x-1}{\sigma_1^3}\phi\left(\frac{x-1}{\sigma_1}\right), p_{\xi}^{''}(x)=-\frac{1}{\sigma_1^3}\phi\left(\frac{x-1}{\sigma_1}\right)+\frac{(x-1)^2}{\sigma_1^5}\phi\left(\frac{x-1}{\sigma_1}\right).$$

$$\max_{x\in(1-\delta,1+\delta)}\left|p_{\xi}^{'}(x)\right| \le \frac{\delta}{\sqrt{2\pi}\sigma_1^3}, \max_{x\in(1-\delta,1+\delta)}\left|p_{\xi}^{''}(x)\right| \le \frac{1}{\sqrt{2\pi}\sigma_1^3}\left(1+\frac{\delta^2}{\sigma_1^2}\right).$$

Using $p_{\eta}(x)=\frac{1}{\sigma_2 x}\phi\left(\frac{\ln x-\mu_2}{\sigma_2}\right)$, we can find

$$p_{\eta}^{'}(x)=-\frac{\sigma_2+d}{x^2\sigma_2^2}\phi(d), p_{\eta}^{''}(x)=\frac{\phi(d(x))}{\sigma_2 x^3}\left(2+\frac{3d(x)}{\sigma_2}+\frac{(d(x))^3-1}{\sigma_2^2}\right),$$

where we designate $d(x)=\frac{\ln x-\mu_2}{\sigma_2}$.

Let us assume that $1-\delta>0$, which will be true in practice as the values of volatilities are usually small. Let us denote

$$d^* = \max_{x\in(1-\delta,1+\delta)} d(x) = \max\left(\frac{\left|\ln(1-\delta)-\mu_2\right|}{\sigma_2}, \frac{\left|\ln(1+\delta)-\mu_2\right|}{\sigma_2}\right).$$

Then,

$$\max_{x\in(1-\delta,1+\delta)}\left|p_{\eta}^{'}(x)\right| \le \frac{\sigma_2+d^*}{\sqrt{2\pi}\sigma_2^2(1-\delta)^2},$$

$$\max_{x\in(1-\delta,1+\delta)}\left|p_{\eta}^{''}(x)\right| \le \frac{1}{\sqrt{2\pi}\sigma_2(1-\delta)^3}\left(2+\frac{3d^*}{\sigma_2}+\frac{(d^*)^3+1}{\sigma_2^2}\right).$$

Combining the obtained inequalities, we find

$$|\Psi| \le \frac{2\delta^3}{3\sqrt{2\pi}N^2}\left[\frac{1}{\sigma_1^3}\left(1+\frac{\delta^2}{\sigma_1^2}\right)+\frac{1}{\sigma_2(1-\delta)^3}\left(2+\frac{3d^*}{\sigma_2}+\frac{(d^*)^3+1}{\sigma_2^2}\right)\right]+$$
$$+\frac{12\delta^2}{\sqrt{2\pi}N^2}\left[\frac{\delta}{\sigma_1^3}+\frac{\sigma_2+d^*}{\sigma_2^2(1-\delta)^2}\right].$$

Calculation of $d_{TV}(\xi,\eta)$ using the Monte Carlo method

The same integral can be calculated using the Monte Carlo method, as

$$\int_{\mathbb{R}}(p_\xi(x)-p_\eta(x))^+dx=\int_{\mathbb{R}}(1-\frac{p_\eta(x)}{p_\xi(x)})^+p_\xi(x)dx=\mathbb{E}[1-\frac{p_\eta(\xi)}{p_\xi(\xi)}]^+,$$

where the expectation is taken with respect to the distribution of a random variable $\xi \sim p_\xi(\cdot)$.
We simulate the independent random variables $X_1,\dots,X_n \sim p_\xi(\cdot)$ and approximate the integral by $\frac{1}{n}\sum_{i=1}^{n}Y_i$, where $Y_i=2(1-\frac{p_\eta(X_i)}{p_\xi(X_i)})^+$. The mean-square deviation in this case can be expressed as

$$\mathbb{E}(\frac{1}{n}\sum_{i=1}^{n}Y_i-d_{TV}(\xi,\eta))^2=Var\left[\frac{1}{n}\sum_{i=1}^{n}Y_i\right]\le\frac{2}{n}$$

Numerical Solution of a Nonlinear Equation

Let us now discuss the numerical solution of (19). Consider the case that has exactly three roots (for cases with fewer roots, the algorithm will be similar). As in the proof of Theorem 3,

$$h(x)=\left(x^2-2x\right)-(\frac{\sigma_1}{\sigma_2})^2(\ln x)^2-3\sigma_1^2\ln x+1-\frac{\sigma_1^2\sigma_2^2}{4}-2\sigma_1^2\ln\left(\frac{\sigma_2}{\sigma_1}\right),$$
$$h'(x)=2(x-1)-(\frac{\sigma_1}{\sigma_2})^2\frac{2\ln x}{x}-\frac{3\sigma_1^2}{x},$$
$$h''(x)=2+(\frac{\sigma_1}{\sigma_2})^2\frac{2(\ln x-1)}{x^2}+\frac{3\sigma_1^2}{x^2}.$$

Equation $h''(x)=0$ has exactly one root, which means that the function $h(z)$ has one inflection point that lies between x_1^* и x_2^* (Fig. 2), and therefore, it is concave on $(0,x_1^*)$ and convex on $(x_2^*,+\infty)$.

This ensures that Newton's method for the root x_1 with an initial point $x_1^{(0)}$ such that $x_1^{(0)}<x_1$ converges to the root. For the same reason, Newton's method will converge to root x_3 at the initial point $x_3^{(0)}>x_3$. Root x_2 can be localized by the bisection method for $[x_1,x_3]$ and then calculated by Newton's method.

According to Samarsky and Gulin (1989), if $h(\cdot)$ is twice continuously differentiable in the neighborhood $U_r(x^*)$ of root x^* of the equation $h(x)=0$, and

$$q=\frac{M_2\left|x^*-x^{(0)}\right|}{2m_1}<1,m_1=\inf_{x\in U_r(x^*)}\left|h'(x)\right|,M_2=\sup_{x\in U_r(x^*)}\left|h''(x)\right|.$$

Then, Newton's method converges to x^*, and

$$\left|x^{(k)}-x^*\right|\le q^{2^k-1}\left|x^{(0)}-x^*\right|.\#(28)$$

Thus, for convergence, it is sufficient to assume that in some neighborhood of the root, the second derivative is bounded and the first one is strictly separated from zero.

At $x>x_1^{(0)}$,

$$M_2 \le 2 + 2(\frac{\sigma_1}{\sigma_2})^2 \max\left(\frac{\left(1-\ln x_1^{(0)}\right)}{(x_1^{(0)})^2}, 1\right) + \frac{3\sigma_1^2}{(x_1^0)^2},$$

and at the localization of the root, the minimum of the modulus of the first derivative is attained at one of the segment endpoints, where it can be computed explicitly. Therefore, by partitioning the segment until $q < 1$, we can achieve a guaranteed rate of convergence (28).

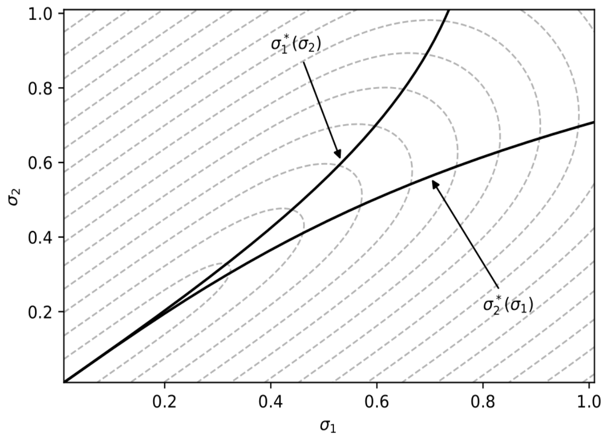

Figure 3. Contour lines $d_{FM}(\xi,\eta)$ and optimum values $\sigma_1^*(\sigma_2), \sigma_2^*(\sigma_1)$ plots

Source: The authors.

Results of Numerical Calculations

The results of metric calculation and optimal values σ_1^*, σ_2^* for the Fortet-Mourier metric at $\sigma_1, \sigma_2 \in (0,1), \mu_1 = 1, \mu_2 = -\frac{\sigma_2^2}{2}$ are presented in Figs. 3 and 4.

The contour lines show that the distances between random variables ξ, η tend to zero as $\sigma_1 \to 0, \sigma_2 \to 0$. This is because of the convergence of distributions ξ, η to the Dirac measure as the volatilities tend to zero.

Application of the Estimates to Certain Options

In this section and hereafter, when referring to processes (1) and (2), we imply that they are martingales; that is, (3) is satisfied.

Estimates (7)–(9), as well as the formulas for the metrics, show that the significant parameters determining the difference between the models are the integrated (or cumulative) volatilities, denoted by σ_1, σ_2.

The application of estimates (7)–(9) to some types of options is shown below.

Put and Call Options

The payoff function of a standard call option $f_C(X_T) = (X_T - K)^+$ is Lipschitz continuous with the Lipschitz constant equal to 1. Therefore, from (7),

$$\left|P_B(f,T) - P_S(f,T)\right| \le d_{FM}\left(X_T^B, X_T^S\right) = X_0 d_{FM}(\xi,\eta).$$

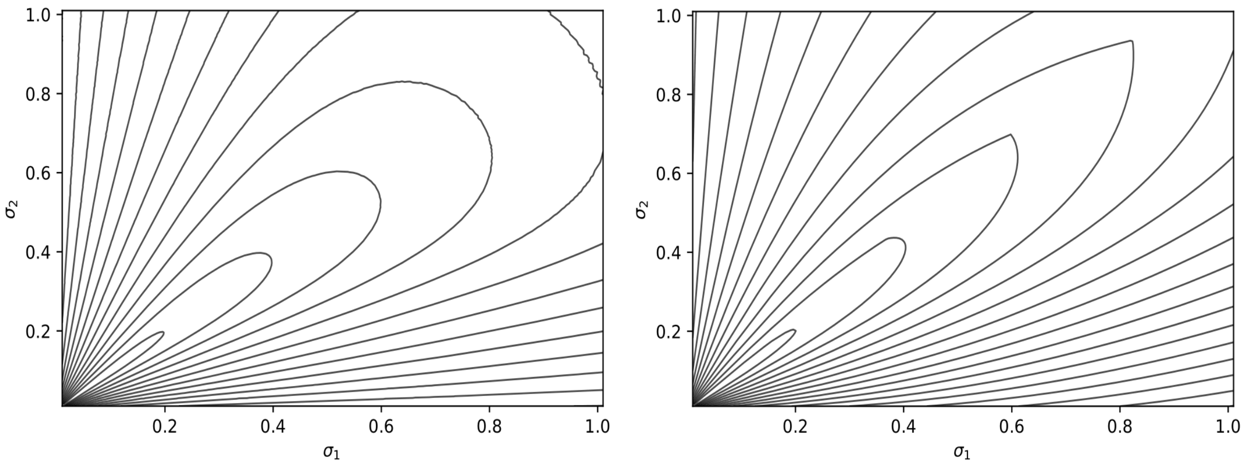

Figure 4. Contour lines $d_{TV}(\xi,\eta)$ and $d_K(\xi,\eta)$

Source: The authors.

Let us use the data obtained by Bachelier (1900). Consider an option with the time to exercise equal to one month, for which the integral volatility equals $\sigma = \sigma_1 = \sigma_2 \approx 0.008$. Then, we find

$$\left|P_B\left(f_C,T\right) - P_S\left(f_C,T\right)\right| \le 3.1\cdot 10^{-5} X_0.\#(29)$$

Exactly the same estimate is true for a put option.

It is also interesting to compare this estimate with that obtained by Schachermayer and Teichmann (2005) for a call option "at the money" (i.e., for $K = X_0$):

$$0 \le P_B\left(f_C,T\right) - P_S\left(f_C,T\right) \le \frac{X_0\sigma^3}{12\sqrt{2\pi}}.$$

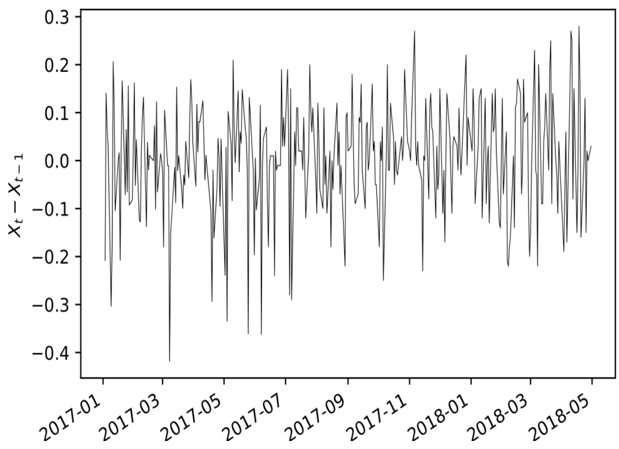

Figure 5. Daily price increments

Source: The authors.

For the same value of σ on the right-hand side, we get $\approx 1.6\cdot 10^{-8} X_0$. Of course, this exceeds the accuracy of (29) by three orders of magnitude; however, the estimation with the Fortet–Mourier metric allows us to work with a very wide class of payoff functions and therefore is a more universal method.

Binary Options

Consider a binary call option with payout function

$$f_{B,C}\left(X_T\right) = M\mathbb{I}_{X_T \ge K}.$$

Then, from (8),

$$\left|P_B\left(f_{B,C},T\right) - P_S\left(f_{B,C},T\right)\right| \le Md_{TV}\left(X_T^B, X_T^S\right).$$

Substituting the Bachelier's data, we obtain

$$\left|P_B\left(f_{B,C},T\right) - P_B\left(f_{B,C},T\right)\right| \le 6\cdot 10^{-3} M.$$

As it was noted, the total variation metric provides less accurate but still acceptable estimate.

Let us also apply (9):

$$\left|P_B\left(f_{B,C},T\right) - P_B\left(f_{B,C},T\right)\right| \le Md_K\left(X_T^B, X_T^S\right) \approx 1.6\cdot 10^{-3} M.$$

The Kolmogorov metric gives a more accurate result, which, however, has the same order as that of the total variation metric.

Estimation of Volatility Using the Oil Market Prices

Let us now try to apply the obtained estimates to the current data. For this purpose, it is necessary to evaluate the parameters σ_B, σ_S of models (1) and (2). Furthermore, we apply statistical estimation methods assuming that the data satisfy the Bachelier model or the Samuelson model. For real market prices, the distribution of their increments or the increments of their logarithms is poorly approximated by the normal distribution and the increments themselves are not independent (e.g., the effect of volatility clusters occurs). These effects are considered using time-series models with conditional heterogeneity (ARCH models) that allow to describe the asset price behavior more precisely. In addition, the processes obtained using these models, with appropriate normalization, converge to diffusion ones (Gouriéroux, 1997; Th. 5.15).

It can justify their application to the estimation of parameters of Bachelier and Samuelson models. However, when comparing these models, we are interested in a rough evaluation of the volatility[9].

Consider price X_t as the closing price for WisdomTree WTI Crude Oil from January 2017 to November 2018 (Figure 5). Let us consider dimensionless values

$$Y_t = \frac{X_t}{X_0}, t = 0,1,\ldots,n = 335.$$

According to the Bachelier model, the price increments $\Delta Y_t = Y_t - Y_{t-1}$ can be represented as

$$\Delta x_t = \alpha + \sigma_B \Delta W_t, \Delta W_t = W_t - W_{t-1} \sim \mathcal{N}(0,1).$$

Thus, as the Wiener process increments are independent, we consider $\{\Delta x_t\}$ as a sample of random variables having a normal distribution $\mathcal{N}(\alpha, \sigma_B^2)$.

The maximum likelihood estimate $\hat{\sigma}_B$ for the standard deviation from the sample obtained from the Gaussian distribution with two unknown parameters, mathematical expectation and variance, is

$$\hat{\sigma}_B = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(\Delta Y_t - \overline{\Delta Y_t})^2},$$

where

$$\overline{\Delta Y_t} = \frac{1}{n}\sum_{t=1}^{n}\Delta Y_t.$$

This estimate gives an approximate value for the volatility $\hat{\sigma}_B \approx 0.0144$.

In the Samuelson model, the logarithm increments

$$\Delta(\ln Y_t) = \gamma + \sigma_S \Delta W_t \sim \mathcal{N}(\gamma, \sigma_S^2).$$

Estimating the standard deviation similarly, we obtain $\hat{\sigma}_S \approx 0.0150$.

Let us construct a confidence interval for the obtained estimates with confidence level q. For the sample $Z_1,\ldots,Z_n$ obtained from normal distribution with two unknown parameters, the mathematical expectation μ and variance σ^2, the random variable $\sum_{i=1}^{n}\frac{(Z_i - \overline{Z}_n)^2}{\sigma^2}$ has a distribution of $\chi^2(n-1)$ (e.g., DeGroot and Schervish (2011)). Therefore, to estimate the maximum likelihood $\hat{\sigma}$ of the scale parameter σ, we have

$$\mathbb{P}\left(\gamma_1 < n\frac{\hat{\sigma}^2}{\sigma^2} < \gamma_2\right) = \chi_{n-1}(\gamma_2) - \chi_{n-1}(\gamma_1) = q,$$

where $\chi_{n-1}(\cdot)$ denotes the cumulative distribution function for the law $\chi^2(n-1)$. Let us choose

$$\gamma_1 = \chi_{n-1}^{-1}\left(\frac{1-q}{2}\right), \gamma_1 = \chi_{n-1}^{-1}\left(\frac{1+q}{2}\right),$$

then the corresponding confidence interval for σ is

$$\left[\hat{\sigma}\sqrt{\frac{n}{\gamma_2}},\hat{\sigma}\sqrt{\frac{n}{\gamma_1}}\right].$$

For the confidence level $q = 0.99$, we obtain the confidence intervals as follows:

$$\sigma_B \in [0.0131, 0.0160], \sigma_S \in [0.0136, 0.0166].\#(30)$$

The obtained results are consistent with the normalized values of Chicago Board Options Exchange (CBOE) Oil Volatility Index (OVX) over the same period of time (Fig. 6). This index is calculated similarly to the volatility index (VIX) but uses oil options. The OVX values should be interpreted as implicit volatility (i.e., volatility calculated based on the observed option prices and reflecting appropriate expectations of market volatility behaviour in the next month). By contrast, the estimates derived from the historical data $\hat{\sigma}_B, \hat{\sigma}_S$ reflect the value of realized volatility; therefore, the comparison of these values is not entirely correct. Nevertheless, our goal is to only estimate the order of magnitudes σ_B и σ_S; thus, it is acceptable for a rough evaluation of "engineering character."

Now we apply the estimate (7) to the call option with the time to expiration equal to one month ($T = 30$) and obtain

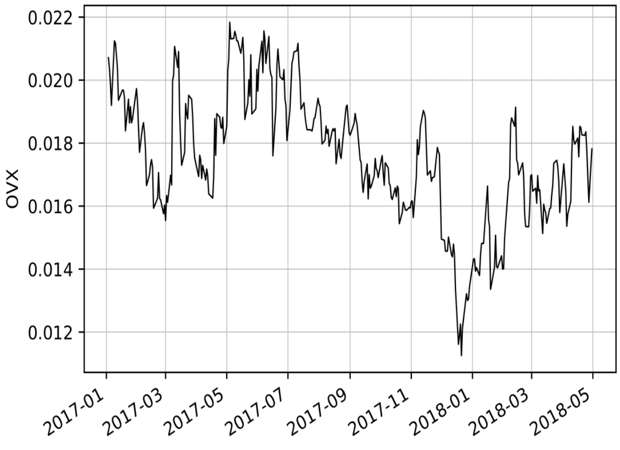

Figure 6. OVX index

Source: The authors.

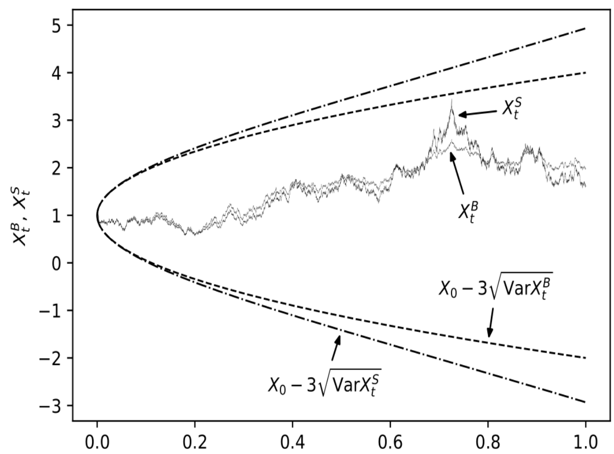

Figure 7. Process trajectories X_t^B, X_t^S.

Source: The authors.

$$\left|P_B(f_C,T) - P_S(f_C,T)\right| \le d_{FM}\left(X_T^B, X_T^S\right) \approx 4.7\cdot 10^{-3} X_0.\#(31)$$

For a binary option with $T = 30$ and payout M, according to (8),

$$\left|P_B(f_B,T) - P_S(f_B,T)\right| \le M d_{TV}\left(X_T^B, X_T^S\right) \approx 7.9\cdot 10^{-2} M.\#(32)$$

If we apply (9), we obtain

$$\left|P_B(f_B,T) - P_S(f_B,T)\right| \le M d_K\left(X_T^B, X_T^S\right) \approx 2.1\cdot 10^{-2} M.\#(33)$$

Values of integral volatility

Let us find at what values of the integral volatility parameter the processes X_t^B, X_t^S remain "close" to each other.

Using the Ito formula (e.g., Øksendal, 1991), we find that X_t^B, X_t^S satisfy the stochastic differential equations

$$dX_t^B = \sigma_B X_0 dW_t,$$

$$dX_t^S = \sigma_S X_t^S dW_t,$$

where for a small t value, the optimal relation between the volatilities is $\sigma_B \approx \sigma_S$.

Let us now calculate the variances:

$$VarX_t^B = X_0^2\sigma_B^2 t, \sqrt{VarX_t^B} = X_0\sigma_B\sqrt{t},$$
$$VarX_t^S = X_0^2\left(e^{\sigma_S^2 t}-1\right), \sqrt{VarX_t^S} = X_0\sqrt{\left(e^{\sigma_S^2 t}-1\right)}.$$

The variances and standard deviations depend only on the initial price and integral volatility. Assuming $X_0 = 1, \sigma = \sigma_B = \sigma_S = 1$, let us model both processes (Fig. 7) such that they correspond to the same Wiener process W_t. At $t \approx 0.2$, the standard deviations and the processes themselves begin to differ appreciably. This value corresponds to the integral volatility value $\sigma\sqrt{t} \approx 0.45$.

For the options considered in the previous section, the integral volatility is approximately equal to $\hat{\sigma}\sqrt{T} \approx 0.015 \cdot \sqrt{30} \approx 0.082$.

Option Price Sensitivity to Volatility

To validate the above-used estimates (31)–(33), the option price must change insignificantly for small changes in volatility. This requirement is based on the fact that the value σ is never exactly known in the model and its estimation leads to an error when calculating the option price. Let us estimate the sensitivity vega (see Hull, 2012)

$$\mathcal{V} = \frac{\partial P(f,T)}{\partial\sigma}$$

for standard and binary put and call options.

The price of a standard call option in the Bachelier model is calculated as

$$P_B(f_C,T) = (X_0 - K)\Phi\left(\frac{X_0-K}{\sigma_B\sqrt{T}X_0}\right) + \sigma_B\sqrt{T}X_0\phi\left(\frac{X_0-K}{\sigma_B\sqrt{T}X_0}\right).$$

Its derivation has been provided by Schachermayer and Teichmann (2005). Similarly, the price of a standard put option can be determined:

$$P_B(f_P,T) = (K - X_0)\Phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}X_0}\right) + \sigma_B\sqrt{T}X_0\phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}X_0}\right).$$

Let us find the vega coefficient for these options:

$$\frac{\partial P_B(f_C,T)}{\partial\sigma_B} = (X_0-K)\phi\left(\frac{X_0-K}{\sigma_B\sqrt{T}X_0}\right)\left(-\frac{X_0-K}{\sigma_B^2\sqrt{T}X_0}\right) + \sqrt{T}X_0\phi\left(\frac{X_0-K}{\sigma_B\sqrt{T}X_0}\right) +$$
$$+\sigma_B\sqrt{T}X_0\phi'\left(\frac{X_0-K}{\sigma_B\sqrt{T}X_0}\right)\left(-\frac{X_0-K}{\sigma_B^2\sqrt{T}X_0}\right) = X_0\sqrt{T}\phi\left(\frac{X_0-K}{\sigma_B\sqrt{T}X_0}\right),$$

as $\phi'(x) = -x\phi(x)$.

Similarly, for a put option,

$$\frac{\partial P_B(f_P,T)}{\partial\sigma_B} = X_0\sqrt{T}\phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}}\right) = \frac{\partial P_B(f_C,T)}{\partial\sigma_B}. \# (34)$$

In the Samuelson model, the prices of standard put and call options are determined using the Black-Scholes formulas:

$$P_S\left(f_C,T\right)=X_0\Phi\left(\frac{\ln\frac{X_0}{K}+\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right)-K\Phi\left(\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right),$$

$$P_S\left(f_P,T\right)=-X_0\left[1-\Phi\left(\frac{\ln\frac{X_0}{K}+\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right)\right]+K\left[1-\Phi\left(\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right)\right].$$

The derivatives of these quantities obtained by σ_S are found to coincide. Denoting

$$y_+=\frac{\ln\frac{X_0}{K}+\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}},y_-=\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}},$$

let us find

$$\frac{\partial P_S\left(f_C,T\right)}{\partial\sigma_S}=\frac{\partial P_S\left(f_P,T\right)}{\partial\sigma_S}=X_0\phi\left(y_+\right)\left(-\frac{\ln\frac{X_0}{K}}{\sigma_S^2\sqrt{T}}+\frac{1}{2}\sqrt{T}\right)-K\phi\left(y_-\right)\left(-\frac{\ln\frac{X_0}{K}}{\sigma_S^2\sqrt{T}}-\frac{1}{2}\sqrt{T}\right).\#(35)$$

For binary call and put options with payout features,

$$f_{B,C}\left(X_T\right)=M\mathbb{I}_{X_T>K},f_{B,P}\left(X_T\right)=M\mathbb{I}_{X_T<K}.$$

Accordingly, the price is determined as an expectation with respect to the martingale measure:

$$P_B\left(f_{B,C}\right)=\mathbb{E}^Bf_{B,C}\left(X_T\right)=M\mathbb{P}^B\left(X_T>K\right)=M\left(1-\Phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}X_0}\right)\right),$$

$$P_B\left(f_{B,P}\right)=M\Phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}X_0}\right),$$

$$P_S\left(f_{B,C}\right)=\mathbb{E}^Sf_{B,C}\left(X_T\right)=M\mathbb{P}^S\left(X_T>K\right)=M\Phi\left(\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right),$$

$$P_S\left(f_{B,P}\right)=M\Phi\left(\frac{\ln\frac{K}{X_0}+\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right).$$

From this, we find

$$\frac{\partial P_B\left(f_{B,C},T\right)}{\partial\sigma_B}=-\frac{\partial P_B\left(f_{B,P},T\right)}{\partial\sigma_B}=M\phi\left(\frac{K-X_0}{\sigma_B\sqrt{T}X_0}\right)\left(\frac{K-X_0}{\sigma_B^2\sqrt{T}X_0}\right),\#(36)$$

$$\frac{\partial P_S\left(f_{B,C},T\right)}{\partial\sigma_S}=-\frac{\partial P_S\left(f_{B,P},T\right)}{\partial\sigma_S}=M\phi\left(\frac{\ln\frac{X_0}{K}-\frac{1}{2}\sigma_S^2T}{\sigma_S\sqrt{T}}\right)\left(-\frac{\ln\frac{X_0}{K}}{\sigma_S^2\sqrt{T}}-\frac{1}{2}\sqrt{T}\right).\#(37)$$

Let us now estimate the order of the price calculation error that appears due to an inaccurate measure of volatility. This error approximately equals to $|\mathcal{V}\Delta\sigma|$, where $\mathcal{V}$ is the option vega coefficient and $\Delta\sigma$ is the volatility measurement error. As options «at the money» have the greatest liquidity, their study is of the greatest interest. Therefore, we further assume that $K = X_0, T = 30$. From (34) and (35) considering confidence intervals (30), we obtain that for the standard options with the confidence probability, equal to 0.99, the error approximation of $\left|P_B(f_C,T)-P_S(f_C,T)\right|$ calculation does not exceed

$$\sqrt{\frac{T}{2\pi}}X_0\max\left|\Delta\sigma_B\right|+\sqrt{T}\phi\left(\frac{1}{2}\hat{\sigma}_S\sqrt{T}\right)X_0\max\left|\Delta\sigma_S\right|\approx 7\cdot 10^{-3}X_0$$

For binary options with $K = X_0, T = 30$, according to (36) and (37), with confidence probability 0.99, the error approximation does not exceed

$$\frac{1}{2}M\sqrt{T}\phi\left(\frac{1}{2}\hat{\sigma}_S\sqrt{T}\right)\max\left|\Delta\sigma_S\right|\approx 1.8\cdot 10^{-3}M.$$

The resulting estimates differ from (31)-(33) by no more than an order of magnitude. Thus, with the estimation methods used, the error associated with an inaccurate measurement of the volatility can make almost the same contribution to the option price as a model change.

In this section, sensitivity estimation is obtained only for the options of a special form. When applying similar methods for classes of functions, the accuracy of the estimation deteriorates considerably. Let us estimate the vega coefficient in the Bachelier model: if we denote $p(\cdot)$ as the density of the random variable $\frac{X_T}{X_0}$, then the price of the European option with payout function $f(\cdot)$ and time to expiration T can be found as $P_B(f,T)=\int_{-\infty}^{\infty} f(yX_0)p(y)dy.$

Based on (1) and (3), the function $p(\cdot)$ can be expressed as $p(y)=\frac{1}{\sigma_B\sqrt{T}}\phi\left(\frac{y-1}{\sigma_B\sqrt{T}}\right).$

After changing the variables $z=\frac{y-1}{\sqrt{T}}$, we obtain

$$P_B(f,T)=\int_{-\infty}^{\infty} f\left(\left(1+\sqrt{T}z\right)X_0\right)\frac{1}{\sigma_B}\phi\left(\frac{z}{\sigma_B}\right)dz.$$

Let us differentiate the integral by parameter σ_B. The differentiation performed under the integral is possible for all $\sigma_B > 0$, as, considering σ_B on each finite interval, the function $\left(f\frac{\partial p}{\partial\sigma_B}\right)(\cdot)$ will be majorized by an integrable function that does not depend on σ_B.

$$\frac{\partial P_B}{\partial\sigma_B}(f,T)=\int_{-\infty}^{\infty} f\left(\left(1+\sqrt{T}z\right)X_0\right)\left[-\frac{1}{\sigma_B^2}\phi\left(\frac{z}{\sigma_B}\right)+\frac{z^2}{\sigma_B^4}\phi\left(\frac{z}{\sigma_B}\right)\right]dz= \quad \#(38)$$
$$=\frac{1}{\sigma_B}\int_{-\infty}^{\infty} f\left(\left(1+\sqrt{T}\sigma_B z\right)X_0\right)\left[-\phi(z)+z^2\phi(z)\right]dz.$$

For a bounded function $f(\cdot)\in B(\mathbb{R})$,

$$\left|\frac{\partial P_B}{\partial\sigma_B}(f,T)\right|\le\frac{\|f\|_B}{\sigma_B}\int_{-\infty}^{\infty}\left[\phi(z)+z^2\phi(z)\right]dz=\frac{2}{\sigma_B}\|f\|_B\ .\#(39)$$

For the Lipschitz continuous functions, we will use the inequality

$$\left|f(x)\right| \le \left|f(X_0)\right| + \| f \|_{Lip} \left|x - X_0\right|.$$

Considering that

$$\int_{-\infty}^{\infty} |x| \phi(x) dx = \frac{2}{\sqrt{2\pi}}, \int_{-\infty}^{\infty} |x|^3 \phi(x) dx = \frac{4}{\sqrt{2\pi}},$$

$$\left|\frac{\partial P_B}{\partial \sigma_B}(f,T)\right| \le \frac{1}{\sigma_B} \int_{-\infty}^{\infty} \left(\left|f(X_0)\right| + \| f \|_{Lip} \sqrt{T} \sigma_B |z| X_0\right) \left[\phi(z) + z^2 \phi(z)\right] dz \le$$
$$\le \frac{2\left|f(X_0)\right|}{\sigma_B} + \frac{6}{\sqrt{2\pi}} \sqrt{T} X_0 \| f \|_{Lip} \qquad \#(40)$$

According to estimates (39) and (40), as well as the confidence interval (30), the calculation error $P_B(f,T)$ for a standard call (put) option in money with $T = 30$ does not exceed

$$\frac{6}{\sqrt{2\pi}} \sqrt{30} \max\left|\Delta\sigma_B\right| X_0 \approx 2 \cdot 10^{-2} \cdot X_0,$$

and for a binary option with $K = X_0, T = 30$ does not exceed

$$\frac{2}{\hat{\sigma}_B} M = 0.22 \cdot M.$$

The resulting accuracy estimates are inferior to those obtained using the exact representation of the vega coefficient for these options by one or two orders of magnitude, which is expected as a consequence of the universality of the estimates.

5 Conclusion

The approach based on the use of probability metrics enables the estimation of how much the transition from one model to another affects the price of a European option with a payout function from a certain class (represented as a sum of Lipschitz continuous and bounded functions). This price change can be estimated by using an appropriate probabilistic metric and the norm (or semi-norm) of the payout function in a suitable function space. However, the main factor affecting the value of the estimation is the integral volatility, at a large value of which the Bachelier and Samuelson models, which are essentially arithmetic and geometric random walks, cease to be similar. As expected, the estimates obtained using the Fortet-Mourier metric were the most accurate, whereas the use of the total variation metric and the Kolmogorov metric led to similarly less accurate results. Moreover, the calculation of the latter two metrics was reduced to the numerical solution of the same nonlinear equation describing the points of intersection of normal and lognormal densities.

For the oil market, measures of realized volatility were estimated and confidence intervals were constructed assuming that the models are true. By calculating the sensitivity (vega coefficient) for standard and binary options, the error arising in the estimation of model parameters was found to be comparable to the change in price when the model changed.

Acknowledgments

The authors express their gratitude to I. Vorobieva for valuable comments and advice in the design of this article, as well as to the reviewer for useful comments, which helped to improve the presentation.

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Footnotes

[1] Apparently, Samuelson was the first economist to propose this modification of the Bachelier model. Therefore, we use the term "Samuelson's model."

[2] For a complete list of contracts, see CME Group Advisory Notice 20–171, 2020.

[3] This follows directly from the Ito formula.

[4] The assumptions made in Bachelier's thesis (in an informal way) actually mean that the price process is a martingale.

[5] The term "coupling" is also used in random process theory in a different sense; see, for example, Sverchkov, and Smirnov (1990).

[6] The generalized inverse function defined in this manner is also left-continuous. In this case, the random variable $F^{-1}(U)$, where U is uniformly distributed on $(0,1)$ random variable, has a distribution function equal to F.

[7] This metric forms the basis of the nonparametric criterion of the same name, which is based on the theorem proved by Kolmogorov (1933).
[8] Also, Kantorovich metric, Wasserstein metric, and Dudley metric. The variety of names can be explained by many equivalent representations (for details, see Rüschendorf, https://wwwhttps://www.encyclopediaofmath.org/index/index.php?title=Wasserstein_metric=Wasserstein_metric).
[9] An exposition of the statistical analysis concerning volatility has been presented by Melnikov, Volkov, and Nechaev (2001), paragraph 4.3. In contrast to this study, we use the maximum likelihood estimation (instead of an unbiased estimation with uniformly minimal variance) for the volatility, as such estimation for bijective transformation of the parameter reduces to this transformation of the parameter estimate. Among other things, this is applicable when determining implicit volatility.

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The article was submitted on 17.05.2021, reviewed on 16.07.2021, and accepted for publication on 17.08.2021. The authors read and approved the final version of the manuscript.
Статья поступила в редакцию 17.05.2021; после рецензирования 16.07.2021; принята к публикации 17.08.2021.
Авторы прочитали и одобрили окончательный вариант рукописи.

ORIGINAL PAPER

DOI: 10.26794/2308-944X-2021-9-3-77-93
JEL C6, C61, C63, G1, G2, G12, G13, G22

On Market Completions Approach to Option Pricing

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ABSTRACT

Option pricing is one of the most important problems of contemporary quantitative finance. It can be solved in complete markets with non-arbitrage option price being uniquely determined via averaging with respect to a unique risk-neutral measure. In incomplete markets, an adequate option pricing is achieved by determining an interval of non-arbitrage option prices as a region of negotiation between seller and buyer of the option. End points of this interval characterise the minimum and maximum average of discounted pay-off function over the set of equivalent risk-neutral measures. By estimating these end points, one constructs super hedging strategies providing a risk-management in such contracts. The current paper analyses an interesting approach to this pricing problem, which consists of introducing the necessary amount of auxiliary assets such that the market becomes complete with option price uniquely determined. One can estimate the interval of non-arbitrage prices by taking minimal and maximal price values from various numbers calculated with the help of different completions. It is a dual characterisation of option prices in incomplete markets, and it is described here in detail for the multivariate diffusion market model. Besides that, the paper discusses how this method can be exploited in optimal investment and partial hedging problems.

Keywords: option pricing; complete markets; incomplete markets; non-arbitrage prices; hedging strategies; risk-management

For citation: **Vasilev, I., Melnikov, A. (2021). On market completions approach to option pricing. *Review of Business and Economics Studies, 9*(3),77-93. DOI: 10.26794/2308-944X-2021-9-3-77-93**

ОРИГИНАЛЬНАЯ СТАТЬЯ

О методе рыночных пополнений в задачах оценки стоимости опционов

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АННОТАЦИЯ

Задача оценки стоимости опционов является одной из самых важных в области современных математических финансов. В случае полного рынка стоимость опциона, исключающая арбитраж, может быть определена единственным образом посредством усреднения по единственной риск-нейтральной мере. Для неполного рынка, однако, риск-нейтральная мера не уникальна и возможно оценить стоимость опциона в виде интервала цен, не допускающих арбитраж, которые были бы приемлемы как для продавца, так и для покупателя контракта. Граничные точки такого интервала характеризуют минимальную и максимальную стоимость, на множестве эквивалентных риск-нейтральных мер данного рынка, а также средние стоимости дисконтированной функции выплаты опциона. Зная границы полученного интервала, в целях риск-менеджмента, инвестор формирует супер-хеджирующие стратегии. В настоящей работе приводится оригинальный подход к решению проблемы оценки границ безарбитражной стоимости опциона на неполном рынке. Суть подхода заключается в добавлении необходимого числа вспомогательных активов с целью получения полного рынка, на котором задача имеет единственное решение. Рассматривая все-

возможные пополнения, возможно также оценить минимальную и максимальную стоимости опционов на неполном рынке и получить интервал безарбитражных цен. Такое описание является дуальной характеристикой интервала стоимости опциона на неполном рынке. Авторы детально рассмотрели применение данного подхода к многомерной диффузионной модели рынка и обсудили возможность применения данного подхода при решении задач неполного хеджирования и оптимального инвестирования.
Ключевые слова: ценообразование опционов; полные рынки; неполные рынки; неарбитражные цены; стратегии хеджирования; управление рисками

1 Introduction

The problem of option pricing remains one of the most attractive and valuable problems. Mathematically, this problem admits a perfect solution if the market is complete, i.e., every contingent claim is attainable in the class of self-financing strategies or, equivalently, only one risk-neutral measure exists. Averaging over such a measure leads to a unique option price, called fair price in such a market. In an incomplete market, where non-attainable contingent claims exist, the situation is much more complicated because there are infinitely many risk-neutral measures. Averaging given discounted contingent claim over each such measure, one can get the whole interval of non-arbitrage option prices in contrast to one price in a complete market. So, in incomplete markets, to solve the option pricing problem, one needs to calculate the end points of this interval or provide their estimates.

In the present paper, we describe a fruitful method of solving the problem mentioned above. The leading idea of the proposed method is to transform the initial incomplete market model in such a way to make it complete and, hence, make it possible to calculate the unique price for a given contingent claim. Further, considering all possible transformations of the initial model, we get a set of non-arbitrage option prices similar to the set that existed in the classical approach. These findings lead to the dual characterisation of this set via minimal and maximal values as lower and upper option prices. Such a method of market completions was independently proposed for different incomplete market models: Karatzas (1997) — for multivariate diffusion models, Melnikov and Feoktistov, (2001) and also Appendix 3 of Melnikov (1999) — for multinomial markets. The approach also works for pricing American options too (see, Guilan, 1999). Since that time, option pricing theory was tremendously developed in different aspects, including imperfect hedging, utility-indifference pricing, etc. It is pretty natural to expand the range of its applications.

We demonstrate that instead of using a set of equivalent local risk-neutral measures as a parameter for fair price interval estimation, an agent can work with an easier-to-interpret set of possible completion assets. For obvious reasons, this approach opens a way to nice flexibility of auxiliary assets and greater practical application as one can potentially find necessary assets to complete the market.

The method of market completions can mainly be used in two different ways. The first approach consists in the *estimation of the price intervals*. As there is a set of possible orthogonal completions available, one may aim at the estimation of the intervals of optimal prices that can be uniquely calculated in complete markets. The second approach is to *pick particular completion*. This idea is similar to choosing a specific measure of risks such as Esscher measure or Minimal Relative Entropy measure (see, for example, Miyahara, 1995). The second approach allows us to be more specific regarding assets required for the market to be complete. In some cases, it might be even possible to reverse-engineer such auxiliary assets, for instance, with the help of the BSDE technique (see Kobylanski, 2000).

In addition to option pricing problems, investors are also interested in finding an optimal strategy in incomplete market, often with some constraints. So, it is natural to look towards applying the proposed dual characterisation for these types of problems. There is a well-developed study in the area of partial hedging in complete markets. In Föllmer and Leukert (1999) and Spivak and Cvitanic (1999), authors considered quantile hedging, or maximisation of the probability of successful perfect-hedging, in Föllmer and Leukert (2000), authors also investigated shortfall minimisation in line with its utility-weighted value minimisation. These articles lay a foundation of partial hedging with the help of Neyman-Pearson lemma and Convex optimisation methods. Since recently, risk exposure is measured with the help of special measures widely used by market participants: Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). The latter one is better known as Expected Shortfall (ES) and was recommended in 2016 in The Market Risk Framework of Basel III — an international regulatory accord. These measures spark a particular interest in their application in the optimal partial-hedging problem. Melnikov & Smirnov (2012) show that it is still possible to apply Neyman-Pearson lemma to CVaR optimisation. Recent papers Cong et al. (2014), Li and Xu (2013), Capinski (2014), and Godin (2015) demonstrate a growing interest in CVaR optimisation. We will demonstrate how the method of market completions becoming a useful tool when solving this type of problems on an incomplete market.

The rest of the paper is structured as follows: Section 2 provides necessary details regarding the model under consideration. With the understanding of the reasons for market structural incompleteness, we move on to the central part of the paper — introducing the Method of Market Completions, which is discussed in Section 3 in line with its comparison to classical methodologies risk-neutral price interval estimation on the incomplete market. Section 4 elaborates on connections between market completions and some alternative methods used for handling market incompleteness. Finally, we briefly cover potential further steps towards solving famous partial hedging problems on the incomplete market in Section 5 and conclude the paper in Section 6.

2 Multivariate Diffusion Market Model

To demonstrate results that follow, we will work with the *Standard Multidimensional Market Model,* which is defined as $(B,S)=(B_t,S_t^1,\dots,S_t^n)_{t\le T}$, where $(B_t)_{t\le T}$ represents the value process of a risk-free asset that is usually assumed to be a bank account and $S_t=(S_t^1,\dots,S_t^n)_{t\le T}$ is a n-dimensional vector process that describes the prices of n risky assets:

$$dB_t = B_t r_t dt, \quad B_0 = 1$$

$$dS_t^i = S_t^i\left(\mu_t^i dt + \sum_{j=1}^{k}\sigma_t^{ij} dW_t^j\right) \tag{1}$$

We will also call $\Sigma_t = \left\{\sigma_t^{ij}\right\}_{i,j}$ a volatility matrix of this model. Note that elements of a k-dimensional vector $W=\left(W^1,\dots,W^k\right)$ are independent standard Brownian motions. In general, one can define a multidimensional market model so that each risky asset price is governed by its own separate Brownian motions that are mutually correlated. However, it was shown, for example, in Dhaene et al. (2013), that both mentioned models are equivalent. Further in this paper, we will use the model with independent "underlying" Brownian motions for illustration.

Let us call the $(\mathcal{F})_{t\le T}$-measurable process $\pi=(\beta_t,\pi_t^1,\dots,\pi_t^n)_{t\le T}$ a *portfolio* (*strategy*). This process would reflect amounts of corresponding assets possessed by an investor. Obviously, the capital or value of such a portfolio can be described as

$$V_t^\pi = \beta_t B_t + \sum_{i=1}^{n}\pi_t^i S_t^i. \tag{2}$$

Note that not all strategies would be appropriate for the investor. Typically, the agent on the market has an initial budget x, and the natural constraint is that strategy value should not fall below some threshold at any moment t while strategy is in action. To accommodate this condition, denote the class of *admissible* portfolios with initial capital x as

$$\mathcal{A}(x)=\left\{\pi : V_0^\pi = x, \exists K(\pi)\ge 0 \text{ s.t. } V_t^\pi \ge -K, \forall t\le T\right\}.$$

For simplicity, we might consider $K=0$, meaning that the investor does not want his portfolio to have negative value at any moment until the maturity of the strategy.

Admissible strategy π is called *self-financing* if the following conditions hold:

$$\int_0^T\sum_{i=1}^{n}\left(\left|\pi_t^i\mu_t^i\right| + (\pi_t^i)^2\sum_{j=1}^{k}(\sigma_t^{ij})^2\right)dt < \infty$$

$$V_t^\pi = V_0^\pi + \sum_{i=1}^{n}\int_0^t \pi_s^i dS_s^i. \tag{3}$$

In other words, strategy is called self-financing if its capital changes only due to changes in asset prices without additional injections or extractions of capital by the investor. We will denote the class of self-financing strategies with initial capital x as $SF(x)$.

Definition 0.1: *Model is called arbitrage-free if there is no strategy $\pi \in SF(x)$ such that it has zero initial cost of investment and leads to non-zero profit at maturity with positive probability:*

$$V_0^\pi = 0, \quad P(V_T^\pi > 0) > 0.$$

It is well known that the market model is arbitrage-free if and only if there exists an equivalent martingale measure. It was shown in Karatzas and Shreve (2000) that for the Standard Multidimensional Market Model (1), the no-arbitrage condition could be summarised in the following proposition.

Proposition 0.1: *If there exists a $(\mathcal{F}_t)_{t\le T}$ – progressively measurable process $\theta = (\theta_t^1, \ldots, \theta_t^k)_{t\le T}$ that satisfies*

$$\sum_{j=1}^{k} \sigma_t^{ij}\theta_t^j = \mu_t^i - r, \quad i = 1,\ldots,n, \quad P - a.s. \tag{4}$$

and

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \sum_{j=1}^{k} (\theta_t^j)^2 \, dt\right)\right] < \infty, \tag{5}$$

then the (B,S) the market is arbitrage free.

In other words, the market is arbitrage-free if system (4) has the solution.

Remark: *The inverse Proposition 0.1 is, in general, not true. Condition (4) should hold. However, Novikov condition (5) is sufficient but not a necessary one for uniform integrability of Girsanov exponent and, consequently, for equivalence of corresponding risk-neutral measure.*

Remark: *Solution to the system (4): θ_t is, actually, the one to use for the famous Girsanov theorem to switch to equivalent risk-neutral measure under which discounted risky assets in the model (1) become martingales.*

Remark: *Condition (4) can be equivalently written in a vector form:*

$$\Sigma_t \theta_t = \mu_t - \boldsymbol{r}$$

where $\theta_t \in \mathbb{R}^k$; $\mu_t, \boldsymbol{r} \in \mathbb{R}^n$ $\forall t \in [0,T]$.

Denoting $\| \sigma_t^i \| = \sqrt{\sum_{j=1}^{k} (\sigma_t^{ij})^2}$, condition (5) can also be written as:

$$\mathbb{E}\left[exp\left(\frac{1}{2}\int_0^T \| \sigma_t^i \|^2 \, dt\right)\right] < \infty,$$

Market Completeness

Definition 0.2: *(*Market completeness*) The market is called* complete *if for any $\mathcal{F}_T$ – measurable payment function $H = H_T(\omega) \ge 0$, such that $\mathbb{E}[\mathrm{H}] < \infty$ there exists a strategy $\pi \in SF(x)$ such that* $\mathbb{P}$ – a.s.

$$V_T^\pi(x) = H.$$

Generally speaking, market incompleteness means that sigma algebra $\mathcal{F}_T^S$ generated by risky assets is *smaller* than $\mathcal{F}$ on which contingent claims are defined, namely, $\mathcal{F}_T^S \subset \mathcal{F}$. There might be different reasons for market incompleteness, including, but not limited to:

1. **Structural**: There are more sources of risks on the market than tradeable assets available. In such a case, it is natural to define sigma algebra for claims as the one generated by underlying sources of risk. In the case of model (1), it would be $\mathcal{F}_T^W$.
2. **Informational**: Some investors may have more information regarding the asset price dynamics on the market than others. Typical cases of Large investor were described in Eyraud-Loisel (2019); Follmer and Schweizer (1991).
3. **Due to complex parameters or restrictions**: When parameters of the model become stochastic values (stochastic volatility, stochastic drift, etc.) which are not observable explicitly on the market.

In this paper, we will focus on the structural incompleteness of the market. Condition for such incompleteness in case of (1) was obtained in Karatzas and Shreve (2000) and Dhaene et al. (2013). We summarise them in the following theorem.

Theorem 1: *Standard financial market $\mathcal{M}$ is complete if and only if a number of available stocks n = k, where k is a dimension of underlying Brownian motion.*

Consequently, to have a complete market, we need to have a proper, non-degenerate volatility matrix Σ_t with $n = k$. As market completeness means the existence of a unique martingale measure $\mathbb{P}^*$, the market is complete if system (4) possesses the unique solution $\theta_t \in \mathbb{R}^k$. Girsanov exponential for transition to that unique martingale measure in the multidimensional case will be written in the following form:

$$\frac{dP^*}{dP} = \exp\left\{ -\sum_{i=1}^{n}\int_0^T \theta_t^i dW_t^i - \frac{1}{2}\sum_{i=1}^{n}\int_0^T (\theta_t^i)^2 dt \right\}. \qquad (6)$$

3 Completions of Diffusion Model and Option Pricing

We now move on and introduce the method of market completions which is the main focus of the present paper. First, we formalise the notion of market completion.

As we already noted, markets, in reality, are barely complete, so it is reasonable to find a way to handle market incompleteness. In the previous chapter, we showed that, when speaking about structural incompleteness, such incompleteness for Standard Multidimensional Diffusion market model demonstrated through the volatility matrix which rank is not full. Or, roughly speaking, when the volatility matrix for tradeable assets has a rectangular shape with more columns (sources of risks represented by independent Brownian motions) than rows (risky assets).

In other words, to obtain a complete market that would correspond to the existing incomplete one, it is reasonable to add more "rows" into the volatility matrix under consideration. This idea forms a foundation of the *method of market completions*.

Obviously, "completing" assets should be independent of existing ones and among each other to solve the issue of a non-full rank volatility matrix. Adding them, we obtain a "proper" volatility matrix that corresponds to some complete market where known and well-developed methods can be applied.

Definitions of the Method of Market Completions

Assume the canonical market model (1) with n risky assets for which $n < k$. As always, asset price dynamics is defined on measure space $(\Omega, \mathcal{F}, P)$ equipped with filtration $\mathbb{F}$ generated by k-dimensional Brownian motion. We will call assets that form this incomplete model *primary assets* or *existing assets*.

Denote S^c a $(k-n)$– dimensional $(\mathcal{F}_t)_{t\le T}$ – adapted process $S^c = (S_t^{n+1}, \dots, S_t^k)_{t\le T}$ with the same structure as primary assets:

$$dS_t^i = S_t^i\left(\mu_t^i dt + \sum_{j=1}^{k}\sigma_t^{ij} dW_t^j\right), \quad i = n+1, \dots, k.$$

With the help of newly introduced assets, we can "fix" initially rectangular volatility matrix for a set of existing risky assets σ:

$$\Sigma = \begin{pmatrix} \overbrace{\sigma_t^{1,1} \quad \cdots \quad \sigma_t^{1,k}}^{k\ risks} \\ \vdots \quad \ddots \quad \vdots \\ \sigma_t^{n,1} \quad \cdots \quad \sigma_t^{n,k} \end{pmatrix} = (n \times k)\, matrix \qquad (7)$$

by adding $k-n$ auxiliary assets introduced:

$$\tilde{\Sigma} = \begin{pmatrix} \sigma_t^{1,1} & \cdots & \sigma_t^{1,k} \\ \vdots & \ddots & \vdots \\ \sigma_t^{n,1} & \cdots & \sigma_t^{n,k} \\ & & \\ \sigma_t^{n+1,1} & \cdots & \sigma_t^{n+1,k} \\ \vdots & \ddots & \vdots \\ \sigma_t^{k,1} & \cdots & \sigma_t^{k,k} \end{pmatrix} = (k \times k)\, matrix \qquad (8)$$

Which helps us to arrive at a properly shaped volatility matrix $\tilde{\Sigma}$.

Definition 0.1: *The* $(\mathrm{k}-\mathrm{n})$– *dimensional* $(\mathcal{F}_\mathrm{t})_{\mathrm{t}\le\mathrm{T}}$ – *adapted process* $\mathrm{S}^\mathrm{c} = (\mathrm{S}_\mathrm{t}^{\mathrm{n}+1}, \dots, \mathrm{S}_\mathrm{t}^\mathrm{k})_{\mathrm{t}\le\mathrm{T}}$ *is called a* completion *for the (B, S) market if the resulting volatility matrix* $\tilde{\Sigma}$ *has full rank for all* $\mathrm{t} \le \mathrm{T}$.

Definition 0.2: *A completion* $\overline{\mathrm{S}}^\mathrm{c} = \left(\overline{\mathrm{S}}^{\mathrm{n}+1}, \dots, \overline{\mathrm{S}}^\mathrm{k}\right)$ *is called orthogonal if it satisfies:*

$$S_t^i, \overline{S}_t^j = 0, \text{for all } i = 1, \dots, n;\ j = n+1, \dots, k;\ t \in [0, T]$$

and

$$\overline{S}_t^i, \overline{S}_t^j = 0, \text{for all } i, j = n+1, \dots, k;\ t \in [0, T]$$

Remark: *Operation* $\langle \cdot, \cdot \rangle$ *is taken from the standard martingale theory and represents the* *quadratic covariation of martingale parts of the processes.*

Further in this paper, the set of orthogonal completions will be denoted as $\mathcal{C}^{ort}$. We demonstrate that any market completion can be transformed into an orthogonal form.

Lemma 1: *For any completion* $\mathrm{S}^{\mathrm{c}} \in \mathcal{C}$ *of the* (B,S) *market, it is possible to find an orthogonal completion* $\bar{\mathrm{S}}^{\mathrm{c}} \in \mathcal{C}^{\mathrm{ort}}$.

Proof

It is enough to show that one can always construct orthogonal completion from non-orthogonal assets. It can be accomplished, for example, with the help of a famous Gram-Schmidt method. Our goal is to construct a process $\bar{S}^c = \left(\bar{S}^{n+1},\ldots,\bar{S}^k\right)$ that satisfies the definition above.

To do it, we first define the stochastic logarithm $H^i = (H_t^i)_{t\le T}$:

$$dH_t^i = \frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{k} \sigma_t^{ij} dW_t^j$$

Considering that $i \neq j$, if $H_t^i, H_t^j = 0$ for all $t \in [0,T]$ then S_t^i, S_t^j. On the other hand, if row-vectors σ_t^i and σ_t^j of volatility matrix are orthogonal for $i \neq j$ for all $t \in [0,T]$, then

$$\left\langle H_t^i, H_t^j \right\rangle = \left\langle \begin{array}{c} H_0^i + \int_0^t \mu_s^i ds + \sum_{l=1}^{k} \int_0^t \sigma^{il} dW_s^l, H_0^j + \\ + \int_0^t \mu_s^j ds + \sum_{l=1}^{k} \int_0^t \sigma^{jl} dW_s^l \end{array} \right\rangle = $$

$$= \int_0^t \sum_{l=1}^{k} \sigma_s^{il} \sigma_s^{jl} ds$$

Consequently, to complete the proof, it is enough to show how to construct orthogonal row-vectors $\bar{\sigma}_t^j$ and it would imply orthogonality of assets.

To construct such vectors, we will use the Gram-Schmidt method of orthogonalisation for $\sigma_t^i, i = 1,\ldots,k$:

$$\bar{\sigma}_t^1 = \sigma_t^1,$$

$$\bar{\sigma}_t^i = \sigma_t^i - \sum_{j=1}^{i-1} \alpha_t^{ij} \bar{\sigma}_t^j,$$

for $i = 2,\ldots,k$ with $\alpha_t^{ij} = \frac{\sigma_t^i, \bar{\sigma}_t^j}{\bar{\sigma}_t^j, \bar{\sigma}_t^j}$ for $i, j = 2,\ldots,k; j < i$. It is easy to see that obtained vectors are indeed orthogonal.

Let us also obtain the assets for completion. Defining $\bar{H}^i = (\bar{H}_t^i)_{t\le T}$ for $i = k+1,\ldots,n$ as

$$d\bar{H}^i = \bar{\mu}_t^i + \sum_{l=1}^{n} \bar{\sigma}_t^{il} dW_t^j,$$

with

$$\bar{\mu}_t^1 = \mu_t^1,$$

$$\bar{\mu}_t^i = \mu_t^i - \sum_{j=1}^{i-1} \alpha_t^{ij} \bar{\mu}_t^j,$$

for $i = 2,\ldots,n$. Final completion assets can be obtained from:

$$d\bar{S}_t^j = \bar{S}_t^j d\bar{H}_t^j, \; j \in \overline{k+1,n}$$

Remark: *Orthogonalisation of drift terms for assets in the proof of lemma above plays a rather technical role. In such a form, one would get a much simpler solution for the (4).*

Working with the Set of Orthogonal Completions Instead of ELMM

Let us now demonstrate that working with the set of possible orthogonal completions would be equivalent to working with the set of equivalent local martingale measures (ELMM). As a reminder, an equivalent probability measure is called equivalent (local) martingale measure if discounted risky asset price under such measure is a (local) martingale. We will demonstrate this in case of the problem of estimation of risk-neutral price interval for an initially incomplete market model.

It is well known that in incomplete markets, there are infinitely many ELMMs. Consequently, the risk-neutral price is not unique, and it is more reasonable to speak about the interval of initial fair prices. From the classical martingale approach, it is known that this interval could be described as:

$$\left(\inf_{\tilde{P} \in \mathcal{M}} E^{\tilde{P}} \left[\frac{f_T}{B_T} \right], \sup_{\tilde{P} \in \mathcal{M}} E^{\tilde{P}} \left[\frac{f_T}{B_T} \right] \right)$$

where f_T — contingent claim maturing at time T and $\mathcal{M}$ — set of all ELMMs.

We will demonstrate that fair price interval boundaries obtained with the help of the method of Market Completions coincide with ones from the classical approach. It is straightforward that by "completing" our (B,S) the market we arrive at volatility matrix $\tilde{\Sigma}$ and force the system (4), or, in this case

$$\tilde{\Sigma}_t\theta_t = \mu_t - r,\ P - a.s.$$

to have a unique solution θ.

For this completed market model, there should exist unique equivalent local martingale measure, parametrised with the help of solution $\{\theta_t\}_{t\geq 0}$. As each "completed" volatility matrix corresponds to particular market completion, there is a one-to-one correspondence between the set of ELMM for the initial incomplete model and a set of orthogonal completions.

Lemma 2:

*A. Each completion S^c uniquely defines a single ELMM in the incomplete market. Moreover, for the equivalent orthogonal complete market (obtained using the method of **Lemma 1**), such local martingale measure will be the same.*

B. Each ELMM $\tilde{P}$ in the incomplete market ($\tilde{P} \in \mathcal{M}$) will be a unique ELMM in the associated completed market model. Therefore, the set $\mathcal{M}$ of ELMMs in the incomplete market is equivalent to the set $\mathcal{M}^c$ of unique ELMMs corresponding to each completion of the market.

This beautiful fact allows us to switch analysis from a very abstract class of Equivalent Martingale measures to a class of "completing" assets. The latter is much easier to interpret and also impose different restrictions such as maximal asset volatility or no short selling on the market. For now, let us focus on fair price calculation.

Theorem 2: *In the incomplete $(\mathrm{B,S})$ market, assume that $\mathrm{r}=0$ and let $\bar{\mu}_t^i$ and $\bar{\sigma}_t^i = \left(\bar{\sigma}_t^{i1},\ldots,\bar{\sigma}_t^{ik}\right)$ be as defined in the proof of **Lemma 1** for $\mathrm{i}=1,\ldots,\mathrm{n}$. Let also $\bar{\mathrm{W}}$ be a standard k-dimensional Brownian motion, with the first n elements given by*

$$\bar{W}_t^i = \frac{1}{\bar{\sigma}_t^i}\sum_{j=1}^{k}\overline{\sigma_t}^{ij} W_t^j \tag{9}$$

for $i=1,\ldots,n$, $t\in[0,T]$, where $\bar{\sigma}_t^i = \sqrt{\sum_{j=1}^{k}\left(\overline{\sigma_t}^{ij}\right)^2}$.

Then the upper hedging price can be expressed as

$$C^*\left(f_T,P\right) = \\ = \sup_{\frac{\bar{\mu}_t^i}{\bar{\sigma}_t^i},\, i=n+1,\ldots,k} E^P\left[\exp\left\{\begin{array}{l} -\sum_{i=1}^{k}\int_0^T \frac{\bar{\mu}_t^i}{\bar{\sigma}_t^i} d\bar{W}_t - \\ -\frac{1}{2}\sum_{i=1}^{k}\int_0^T\left(\frac{\bar{\mu}_t^i}{\bar{\sigma}_t^i}\right)^2 dt \end{array}\right\} f_T\left(\bar{W}\right)\right]. \tag{10}$$

Change of Numeraire

In line with the Equivalent Martingale Measure approach, it is also worth mentioning the so-called change of numeraire pricing approach. Its connection to the method of market completions was described in Guilan (1999). We provide the main steps below for informational purposes and to complete an overview of the method of Market Completions in application to pricing problem.

According to this approach, instead of trying to "re-weight" the probability of events by choosing some risk-neutral measure, one is searching for a special portfolio that could be used as discounting factor instead of the classical bank account. However, the choice criteria for such discounting portfolio stays the same — discounted strategy prices should be martingales.

More formally, the main goal is to find a portfolio, which value process X_t is a strictly positive, continuous Ito process such that:

$$dX_t = X_t\left(r_t dt + \pi_t^*\sigma_t\left(dW_t + u_t dt\right)\right).$$

Remark: *Here, we will intentionally use notation u instead of θ just to distinguish approaches. However, they both represent the same idea of the price of the risk.*

We want to use this portfolio as numeraire, such that risk-premiums with respect to this numeraire are constrained to be equal 0. In other words, the price process, discounted by a mentioned portfolio, will be local martingale w.r.t. "objective" probability P.

Theorem 3: *Let $\alpha_t = \left(\sigma_t\sigma_t^T\right)^{-1}\left(\mu_t - r_t 1\right)$, i.e., $u_t = \sigma_t^T \cdot \alpha_t$. Consider the self-financing strategy $\pi_t = (\alpha_t^i)_{i=1}^n$ in the risky assets. Denote by $\mathrm{M_t}$ the present value of this admissible strategy. Then $\mathrm{M_t}$ satisfies SDE:*

$$dM_t = M_t(r_t dt + \left(u_t\right)^T\left(dW_t + u_t dt\right)) = \\ = M_t(r_t dt + \|u_t\|^2\, dt + \left(u_t\right)^T dW_t) \tag{11}$$

In the market with M_t as numeraire, investors are risk-neutral. M-price process $S_t^M = \frac{S_t}{M_t}$ of any asset S_t is a local martingale. We refer to it as a market numeraire.

Proposition 0.1: *If m is a strategy that corresponds to $\mathrm{M_t}$, then:*

- *m maximises the expected logarithm of terminal wealth*
- *m is unique even in an incomplete market*
- *m maximises the expected growth rate.*

Details about mentioned properties can be found in Bajeux-Besnainou and Portait (1997).

Price of European contingent claim f_T on the complete market, according to market numeraire approach could be found as:

$$V_0 = \mathbb{E}^P\left(\frac{f_T}{M_T}\right). \tag{12}$$

When one is working with the incomplete market case, it is obvious that *there are several risk-neutral prices,* as we can find several α_t that fit conditions of Theorem 2. Let us now apply the market completions approach and show that it can estimate option price boundaries on an incomplete market.

Let us consider some market completion S^c. Then coefficients of these fictitious assets satisfy

$$\det\left(\sigma(\rho)\right) = \det\begin{pmatrix}\sigma_t \\ \rho_t\end{pmatrix} \neq 0 \text{ and } u(\rho,a,t) = \begin{pmatrix}\sigma_t \\ \rho_t\end{pmatrix}^{-1}\begin{pmatrix} b_t - r_t I_n \\ a_t - r_t I_{k-n}\end{pmatrix}. \tag{13}$$

with

$$\int_0^T \| u(\rho,a,t)\|^2 \, dt < \infty, \quad P - a.s.$$

On the completed market, one can define market numeraire as in (11):

$$dM(\rho,a,t) = M(\rho,a,t)(r_t dt + \| u(\rho,a,t)\|^2 \, dt + \left(u(\rho,a,t)\right)^T dW_t)$$

In the completed market, we have the fair price of CC f_T calculated similarly to (12):

$$V_0(\rho,a) = \mathbb{E}^P\left(\frac{f_T}{M(\rho,a,T)}\right).$$

Let

$$V_1(\rho) = \inf_{a\in\mathcal{D}_\rho} V_0(\rho,a), \quad V_2(\rho) = \sup_{a\in\mathcal{D}_\rho} V_0(\rho,a)$$

with $\mathcal{D}_\rho = \left\{ a : \mathbb{R}^{k-n} \text{ valued progressively measurable processes such that} \int_0^T \| u(\rho,a,t)\|^2 \, dt < \infty \text{ a.s.} \right\}$.

According to Guilan (1999), the following proposition holds.

Proposition 0.1: *$V_1(\rho)$ and $V_2(\rho)$ are independent of ρ.*

Proposition 0.1 serves as another proof that it is enough to work with orthogonal completions only. Let us pick the orthogonal completion $\sigma\rho^T = 0$, $\rho\rho^T = I$. For such ρ and $a \in \mathcal{D}_\rho$:

$$u(\rho,a) = \begin{pmatrix}\sigma \\ \rho\end{pmatrix}^{-1}\begin{pmatrix}\mu_t - rI_n \\ a_t - rI_{k-n}\end{pmatrix} = \sigma^T\left(\sigma\sigma^T\right)^{-1}\left(\mu - rI_n\right) + \rho^T\left(a - rI_{k-n}\right) = u + \psi = u_\psi. \tag{14}$$

And this u_ψ would be used for the construction of market numeraire. Also, it follows that

$\sigma\psi = 0,$

and

$a = \rho\psi + rI_{k-n}$

It means that the "non-arbitrage" vector u_ψ on the completed market can be decomposed into u from incomplete source market and ψ which is completion dependent. If we define class

$$K(\sigma)=\left\{\psi:\psi \text{ is } \mathbb{R}^k-\text{valued progressively measurable}, \sigma_t\psi_t=0, \forall t\in[0,T], a.s. \text{ and} \int_0^T \|\psi_t\|^2\,dt<\infty, a.s.\right\},$$

then this class will be a parameter space for fictitious completions of the incomplete market. For each $\psi\in K(\sigma)$ one can find a fair price in a completed market. It implies that option price boundaries will be

$$J(t)=\sup_{\psi\in K(\sigma)}\mathbb{E}\left[M_\psi(t)\frac{f_T}{M_\psi(T)}\middle|\mathcal{F}_t\right] \quad\text{or}\quad \inf_{\psi\in K(\sigma)}\mathbb{E}\left[M_\psi(t)\frac{f_T}{M_\psi(T)}\middle|\mathcal{F}_t\right]$$

Remembering results from Guilan (1999), it is possible to show that these price boundaries coincide with boundaries from the classical approach:

$$V(t)=\sup_{\tilde{P}\in\mathcal{M}}\mathbb{E}^{\tilde{P}}\left[\mathrm{B_t}\frac{f_T}{\mathrm{B_T}}\middle|\mathcal{F}_t\right] \quad\text{or}\quad \inf_{\tilde{P}\in\mathcal{M}}\mathbb{E}^{\tilde{P}}\left[\mathrm{B_t}\frac{f_T}{\mathrm{B_T}}\middle|\mathcal{F}_t\right]$$

In other words, it was also shown that $J(t)$ coincides with $V(t)$. For more details, we also encourage the reader to carefully read Guilan (1999) research.

4 Completions in Context of Markov Factors, Dimension Reductions and Jumps

Connection to Markov Factor Model

Denote Girsanov exponential (6) as $\bar{Z}_t=\frac{dP^*}{dP}$. It is known that this process is a solution for

$$d\bar{Z}_t=\bar{Z}_t\bar{\theta}_t dW_t,\quad \bar{Z}_0=1$$

and noting that $\bar{\theta}_t^i=\frac{\bar{\mu}_t^i}{\bar{\sigma}_t^i}$ from non-arbitrage condition $\Sigma_t\bar{\theta}_t=\bar{\mu}_t$ $(r=0)$. Then equation (10) can be re-written in the following form:

$$C^*(f_T,P)=\sup_\theta \mathbb{E}^P\left[Z_T(\theta)f_T(\bar{W})\right].$$

Moreover, as the first k elements of the vector $\bar{\theta}$ are independent of the choice of completion and only depend on the correlations between existing assets, one can represent vector $\bar{\theta}$ as $\bar{\theta}_t=\bar{a}_t+\bar{c}_t,\ \bar{a}_t,\bar{c}_t\in\mathbb{R}^n$ where the first one contains elements of $\bar{\theta}_t$ calculated based on existing assets only. Namely:

$$\bar{a}=[\bar{\theta}_t^1\ldots\bar{\theta}_t^k,0]^T,$$

$$\bar{c}=[0,\bar{\theta}_t^{k+1}\ldots\bar{\theta}_t^n]^T.$$

In this case, the equation for Girsanov exponential can be re-written as

$$dZ_t=Z_t\bar{a}_t dW_t+Z_t\bar{c}_t dW_t,\quad Z_0=1$$

Hence, market completions can be connected to the Markov Factor Model:

$$dS_t=D(S_t)\big(\mu(Y_t)dt+\sigma(Y_t)dW_t\big)$$
$$dB_t=rB_t dt$$

where Y_t is $(k-n)$ dimensional factor process, which does not contain any price processes

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t,$$

μ_Y and σ_Y are vector functions of appropriate dimensions. Or, more conveniently, to the Independent Factor Markov Model in which we assume that vector-valued Wiener process W could be split as

$$\mathrm{W} = \begin{pmatrix} W^S \\ W^Y \end{pmatrix} \quad (15)$$

such that W^S is n-dimensional and corresponds to existing assets on the market and W^Y is $(k-n)$ dimensional and corresponds to factors. In this setting, Markov Factor Model can be written as:

$$dS_t = D(S_t)\big(\mu(Y_t)dt + \sigma(Y_t)dW_t^S\big),$$
$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t^Y,$$
$$dB_t = rB_t dt.$$

It is possible to show that split (15) is similar to what was demonstrated in (9) with existing assets on the incomplete market being assigned, in fact, to $\bar{W}^i$ for $i \in 1..n, t \in [0,T]$ which corresponds to W^S and the rest of $\bar{W}^i$ being assigned to W^Y as it only depends on $(k-n)$ dimensional Brownian motion.

Remark: To briefly demonstrate the idea of transformation completions notation into Markov Factor model one. Assume that we performed the transformation mentioned in (9) for the Standard Multidimensional Diffusion Market model. In this case, it is easy to see that the "completed" volatility matrix can be written as:

$$\tilde{\Sigma} = \begin{pmatrix} L_{n\times n} & 0_{n\times(k-n)} \\ 0_{(k-n)\times n} & D_{(k-n)\times(k-n)} \end{pmatrix}$$

Where $\mathrm{L}_{\mathrm{n\times n}}$ *is a lower triangle matrix and* $\mathrm{D}_{(\mathrm{k-n})\times(\mathrm{k-n})}$ *is a diagonal one. This leads to the natural split of vector W into two parts. Without loss of generality, one might assume first n elements of W to be denoted as* $\mathrm{W}^{\mathrm{S}} \in \mathbb{R}^{\mathrm{n}}$ *and the last (k-n) elements as* $\mathrm{W}^{\mathrm{Y}} \in \mathbb{R}^{\mathrm{k-n}}$.

Dimension Reduction

Another natural approach to transform the volatility matrix into a proper one would be to "trim" it. Or somehow "regroup" underlying Brownian motions in such a way that the reduced volatility matrix for them will have the proper shape. This idea was introduced by Zhang (2007).

For the introduced Standard Multidimensional Diffusion Market Model, dynamics of each risky asset price is governed by the sum of independent standard Brownian motions

$$dS_t^0 = S_t^0 r_t dt, \quad S_0^0 = 1$$
$$dS_t^i = S_t^i\left(\mu_t^i dt + \sum_{j=1}^{k}\sigma_t^{ij}dW_t^j\right), \quad i = 1,\ldots,n.$$

However, as already mentioned, it is possible to write down an equivalent market model which would be governed by n correlated Brownian motions instead of k independent ones (see, e.g., Dhaene et al., 2013):

$$dB_t^i = \sum_{j=1}^{k}\frac{\sigma_t^{ij}}{\|\sigma_t^i\|}dW_t^j$$

then

$$dS_t^i = S_t^i\left(\mu_t^i dt + \|\sigma_t^i\|\, dB_t^i\right), \quad i = 1,\ldots,n.$$

Obviously, obtained Brownian motions are not independent anymore, namely

$$dB_t^i dB_t^j = \rho_t^{ij}$$
$$\rho_t^{il} = \frac{\sum_{j=1}^{k}\sigma_t^{ij}\sigma_t^{lj}}{\|\sigma_t^i\|\cdot\|\sigma_t^l\|}.$$

In this model, we have n-dimensional Brownian motion vector with correlated components $B_t = (B_t^1,\ldots,B_t^n)$, the relationship between which can be described by matrix $\Psi_t = \{\rho_t^{il}\}_{i,l=1..n}$. Notice that Ψ_t is a non-singular, symmetric, and positive semi-definite. That implies the existence of matrix square-root A_t:

$$\Psi_t = A_t \cdot A_t^T, \quad A_t = \{a_t^{ij}\}_{i,j=1..n}$$

Moreover, $\exists \tilde{W}_t^1,\ldots,\tilde{W}_t^n$ independent, such that:

$$B_t^i = \sum_{j=1}^{n}\int_0^t a_s^{ij}d\tilde{W}_s^j.$$

As a result, risky assets can be presented in a form

$$dS_t^i = S_t^i\left(\mu_t^i dt + \sigma_t^i \sum_{j=1}^{n} a_t^{ij} d\tilde{W}_t^j\right), \quad i = 1,\ldots,n.$$

$$\tilde{\sigma}_t = \rho A_t$$

Where Ξ_t is a diagonal matrix of σ^i and matrix A_t depends on the particular decomposition of Ψ_t. According to Zhang (2007), one obtains the following model:

$$dS_t^i = S_t^i\left(\mu_t^i dt + \sum_{j=1}^{n} \tilde{\sigma}_t^{ij} d\tilde{W}_t^j\right), \quad i = 1,\ldots,n.$$

$$\theta_t = A_t^{-1} \cdot \Xi_t^{-1} \cdot \left(\mu_t - r_t 1_n\right)$$

$$\theta_t{}^2 = \left(\mu_t - r_t 1_n\right)^T \cdot \left(\sigma_t \sigma_t^T\right)^{-1} \left(\mu_t - r_t 1_n\right)$$

Completions for the Models with Jumps

The idea of adding auxiliary assets to the market to make it complete is not limited to the diffusion market model. There were also some developments towards a more general geometric Levy model in which asset price is governed by jumps

$$\begin{aligned} &dB_t = rB_t dt \\ &dS_t = S_{t-}\left(\mu dt + dZ_t\right)\right) \quad S_0 > 0, \\ &Z_t = \sigma W_t + X_t \end{aligned}$$

where X_t is a pure jump process and W and X are independent variables. It is well known that such Levy model is not complete even in a one-dimensional case as it includes jumps and Brownian motions as two independent sources of risk and only one asset to use. So instead of introducing the same structure auxiliary assets, authors in Corcuera et al. (2005) enlarge the Levy market with the so-called *i th-power-jump assets* defined as

$$X_t^{(i)} = \sum_{0<s\le t} (\Delta X_s)^i, \quad i \ge 2,$$

where $\Delta X_s = X_s - X_{s-}$ and $X_t^{(1)} = X_t$. Processes $X^{(i)}$ are again Levy processes. These power-jump processes jump at the same time as the original Z_t; however, jump sizes are the i-th power of jumps of the original process. Note, that $X_t^{(i)} = Z_t^{(i)}, i \ge 2$. It is convenient to re-write these assets in the compensated form

$$Y_t^{(i)} = Z_t^{(i)} - E\left[Z_t^{(i)}\right] = Z_t^{(i)} - m_i t, \quad i \ge 1.$$

Enlargement of the model is then consisting in allowing to trade in assets:

$$H_t^{(i)} = e^{rt} Y_t^{(i)}, \quad i \ge 2.$$

With these assets available, it was demonstrated in Corcuera et al. (2005) that any square-integrable martingale M_t can be represented as follows:

$$M_t = M_0 + \int_0^t h_s d\tilde{Z}_s + \sum_{i=2}^{\infty} h_s^{(i)} dY_s^{(i)}$$

where h_s and $h_s^{(i)}, i \ge 2$ are predictable processes such that

$$\tilde{Z} = Z_t - \left(\mu - r\right)t, t \ge 0$$

and

$$E\left[\int_0^t |h_s|^2 ds\right] < \infty$$

$$E\left[\int_0^t |h_s^{(i)}|^2 ds\right] < \infty.$$

In other words, for any square-integrable contingent claim f (non-negative, $\mathcal{F}_T$ measurable random variable) we can set up a sequence of self-financing portfolios whose final values converge in $L^2\left(P^*\right)$. This portfolio will consist of a finite number of bonds, stocks and i th-power-jump assets. It means that f can be replicated, and the market is approximately complete.

This interesting result is important to consider within the general idea of market completion because it offers to search for more specific auxiliary assets beyond just structure-preserving ones discussed before. In the case of the Levy market model or another model with jumps, it might be more convenient to pick specific types of completing assets for each kind of risks presented. It is also useful in terms of interpretation of the auxiliary assets as power-jump-assets are by nature instruments that give exposure to moments like variance (2ndpower-jump asset) or skewedness and kurtosis of distribution (3rd and 4th correspondingly). Assets of such type might be more convenient to introduce to real markets to fix their incompleteness.

5 Completions in Optimal Investment and Partial/Imperfect Hedging

Let us now elaborate more on the application of the method of Market Completions. In this section, we mainly focus on another big part of the area of the Mathematical Finance field — hedging of contingent claims with the major focus on partial hedging.

The idea of introducing fictitious assets to complete the market has already demonstrated potential on the side of partial hedging. First, it is reasonable to look at the classical approaches of partial hedging known for complete market and demonstrate potential towards implementing market completions method for the incomplete case. As it is known, the most up-to-date risk measure approved in the Basel III accord is CVaR.

Definition 0.1: *Value-at-Risk (VaR) measure of a loss* X *can be defined as*

$$VaR_\alpha(X) = \inf a : P(X > a) \le \alpha$$

Definition 0.2: *Conditional Value-at-Risk (CVaR) measure of a loss* X *can be defined as*

$$CVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_\alpha(x)dx,$$

The problem of CVaR optimal hedging contingent claim H under budget constraint $x \le \tilde{V}$ therefore can be stated as

$$\begin{cases} CVaR_\alpha(x,\pi) \to \min\limits_{(x,\pi)} \\ x \le \tilde{V} \end{cases} \tag{16}$$

Inspired by Rockafellar and Urasev, Melnikov and Smirnov in Melnikov and Smirnov (2012) demonstrated that, introducing the special function of parameter $z \in \mathbb{R}$

$$c(z) = z + \frac{1}{1-\alpha}\min_{(x,\pi)}\mathbb{E}\left[(H(z) - V_T^\pi(x))^+\right]$$

where $H(z) = (H - z)^+$ — modified contingent claim H, the problem of CVaR minimisation in case of European contingent claim will be equivalent to the following one:

$$\min_{z\in\mathbb{R}} c(z) = \min_{(x,\pi)} CVaR_\alpha(x,\pi).$$

Consequently, solution of (16) can be decomposed into consequent optimisation by z after solving "internal" problem:

$$\begin{cases} E\left[\left(\mathrm{H}(\mathrm{z}) - \mathrm{V}_\mathrm{T}^\mathrm{p}(\mathrm{x})\right)^+\right] \to \min\limits_{\pi\in\mathcal{A}} \\ x \le \tilde{V} \end{cases} \tag{17}$$

Alternatively, one can approach this problem from the perspective of optimal split into hedged/unhedged proportions of the claim $H = f(H) + R_f(H)$, where $f(H)$ describes the optimal hedged proportion of the claim. This method was offered by Cong et al. (2014).

Considering European type contingent claim, we expect to have a pay-off at maturity time T, so the total *risk exposure* of the investor is going to be

$$T_f(X) = R_f(X) + e^{rT}\Pi(f(X)), \tag{18}$$

where $\Pi(f(X))$ is some chosen pricing functional for the hedged part of exposure.

Given the initial budget constraint, the investor is pursuing the goal of minimising risk measure of total exposure (18), given the restriction on the initial cost of hedging

$$\begin{cases} \min\limits_{f\in\Omega} CVaR(T_f(X)) \\ s.t. \quad \Pi(f(X)) \le \pi_0 \end{cases}$$

According to Cong et al. (2014), under particular assumptions, an explicit way of identifying the optimal hedged loss function is stated in the following theorem.

Theorem 4: *Assume that pricing functional is linear for any time-*t *contingent payout* Z*. Then, the optimal hedged loss function* g_f^* *is given by*

$$g_f^*(x) = (x - d^*)^+ - (x - u^*)^+$$

where (d^*, u^*) *satisfies the following equations*

$$\begin{cases} e^{-rT}\int_{d^*}^{u^*}\mathbb{Q}(X > x)dx = \pi_0 \\ \mathbb{P}(X > u^*) = \alpha \cdot \dfrac{\mathbb{Q}(X > u^*)}{\mathbb{Q}(X > d^*)} \end{cases}$$

and $\mathbb{Q}$ is a risk-neutral measure.

In both approaches, we arrive at some known problem that is well described for the complete market. Consequently, completion of the market can be helpful as it helps to "parametrise" a solution by the set of completing assets. Therefore, choice of the proper completion by market conditions such as partial equilibrium Hu et al. (2005), Esscher measure or Minimal Relative Entropy Measure will be a powerful tool for solving CVaR optimisation problems on the incomplete market.

To demonstrate sensibility of usage of Method of Market Completions for solving stated partial hedging problem (17), we provide existing techniques of partial hedging where Method of Market Completions has already demonstrated great potential or ready to be implemented.

Utility Maximisation

Let us start with the simple case when the goal of the investor is to finance a strategy that provides the greatest terminal wealth utility.

$$v_0(x) \equiv \sup_{\pi\in\mathcal{A}(x)} E\left[U\left(V_T^{\pi}(x)\right)\right].$$

In Karatzas et al. (1991), authors have shown how to obtain such optimal solution with the help of *convex duality methods*. In the core of these methods lies Legendre-Fenchel transform $\tilde{U}(y) \equiv \max_{x>0}\left(U(x) - xy\right) = U\left(I(y)\right) - yI(y)$, where $I : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as the continuous decreasing inverse function of $U'(x)$ (details in Touchette, n.d.).

The solution to this problem for the complete market was given explicitly and can be summarised as the following theorem:

Theorem 5: *For a given initial budget $\tilde{V}_0 > 0$, under the assumption that function*

$$\mathcal{X}_0(y) \equiv E\left[\beta_T Z_T^0 \cdot I\left(y\beta_T Z_T^0\right)\right]\langle\infty, \forall y\rangle 0,$$

the optimal terminal wealth of a strategy can be found as

$$\xi_0^{\tilde{V}_0} = I\left(\mathcal{Y}_0\left(\tilde{V}_0\right)\beta_T Z_T^0\right)$$

where $\mathcal{Y}_0$ is the inverse of the function $\mathcal{X}_0$.

By introducing martingale $X_t \equiv E\left[\beta_T Z_T^0 \xi_0^x | \mathcal{F}_t\right]$ with stochastic integral representation $X_t = \tilde{V}_0 + \int_0^t \varphi_s^T dW_s$ with $\varphi \in \mathcal{F}_t$ and $\int_0^T \|\varphi_s\|^2 ds < \infty$, replicating portfolio for optimal terminal capital can be obtained as

$$\hat{\pi} \equiv \frac{1}{X_t}(\Sigma_t^T)^{-1}\left(\varphi_t + X_t\theta_t\right).$$

Applying the Method of Market Completions for the case of incomplete markets, one can introduce $k-n$ fictitious assets in addition to n existing assets on the incomplete market, driven by the same k-dimensional Brownian motion as n real tradeable assets. Then, the problem of utility maximisation can be solved in the completed market with fictitious assets, but there are infinitely many ways to introduce those fictitious assets.

The relative risk process can then be represented as $\tilde{\theta}_t \equiv \theta_t + \nu_t$ with $\theta_t^T \nu_t = 0$. That means completions could be parametrised by ν which is square-integrable, $\mathcal{F}_t$ adapted and $\mathbb{R}^d$ valued process.

Denote also exponential local martingale:

$$Z_t^{\nu} \equiv \exp\left\{-\int_0^t \tilde{\theta}_s^T dW_s - \frac{1}{2}\int_0^t \left(\theta_s{}^2 + \nu_s{}^2\right)ds\right\}$$

and the function

$$\forall y > 0, \quad \mathcal{X}_\nu(y) \equiv E\left[\beta_T Z_T^\nu I\left(y\beta_T Z_T^\nu\right)\right].$$

Also $\forall \nu \in K_1(\Sigma)$ where

$K_1(\Sigma) \equiv \nu \in K(\Sigma), \mathcal{X}_\nu(y)\langle\infty, \forall y\rangle 0$, define

$$\xi_\nu^x \equiv I\left(\mathcal{Y}_\nu(x)\beta_T Z_T^\nu\right)$$

where $\mathcal{Y}_\nu$ again is the inverse function of $\mathcal{X}_\nu$

An attainable solution will give us value less or equal than that. If we find a strategy π with initial capital x, which does not require the purchase of the artificial stocks and completion $\lambda_t \in K_1(\Sigma)$ such that

$$\sup_{\pi \in \mathcal{A}(x)} E\left[U\left(V_T^{\pi}\right)\right] = E\left[U\left(\xi_{\lambda}^{x}\right)\right],$$

then, for sure, (π, λ) would be optimal. In Karatzas et al. (1991) the following theorem was proven.

Theorem 6: *If we call*

Optimality of π *:* $EU\left(V_T^{\pi}\right) \leq EU\left(V_T^{\hat{\pi}}\right) \quad \forall \pi \in \mathcal{A}(x)$

Financiability of ξ_{λ}^{x} *:* $\exists \hat{\pi} \in \mathcal{A}(x)$ *such that* $V_T^{\hat{\pi}} = \xi_{\lambda}^{x}$

Least Favorability of λ *:*

$$EU\left(\xi_{\lambda}^{x}\right) \leq EU\left(\xi_{\nu}^{x}\right) \quad \forall \nu \in K_1(\Sigma)$$

Parsimony of λ *:* $E\left[\beta_T Z_T^{\nu} \xi_{\lambda}^{x}\right] \leq x, \quad \forall \nu \in K_1(\Sigma)$

Then $B \Leftrightarrow D \Rightarrow C$.

Furthermore, if B holds, then the portfolio $\hat{\pi}$ *in B satisfies A.*

This theorem provides a powerful instrument in verifying if one can build an optimal strategy without artificial assets in use, in other words, when $\lambda = 0$ will satisfy necessary criteria in Theorem 1. It was shown to be the case in Karatzas et al. (1991) for $U(X) = \ln(X)$ and, under some special conditions for $U(X) = \frac{X^{\delta}}{\delta}$ where $\delta < 1, \delta \neq 0$. This edge can further be applied to more specific problems of partial hedging. We provide one example from Karatzas et al. (1991) to demonstrate the application of these conditions to the classical logarithm utility function.

Example 1: *Classical example of Utility function to consider is* $\mathrm{U}(\mathrm{x}) = \ln(\mathrm{x})$. *For this function one has:*

$$\mathcal{X}_{\nu}(y) = \frac{1}{y}, \quad \mathcal{Y}_{\nu}(x) = \frac{1}{x}$$

and optimal terminal capital can be calculated as

$$\xi_{\nu}^{x} = \frac{x}{\beta_T Z_T^{\nu}}.$$

One could check that completion with parameter $\lambda = 0$ *satisfies D.*

$$E\left[\beta_T Z_T^{\nu} \xi_0^{x}\right] = x \cdot E\left[\exp\left\{ -\int_0^T \nu_s^T dW_s - \frac{1}{2}\int_0^T \|\nu_s\|^2 ds \right\}\right] \leq x \quad \forall \nu \in K(\Sigma)$$

as the process under expectation is a supermartingale. It means that investor would not use auxiliary stocks to form an optimal portfolio even for hedging purposes.

Efficient Hedging

One of such problems emerges when given an amount of initial capital v_0 investor's goal is to find the admissible strategy with terminal wealth V_T such that

$$\begin{cases} \mathbb{E}[U\left(\left(H - V_T\right)^{+}\right)] = \min \\ \sup\limits_{P^* \in \mathcal{P}} \mathbb{E}^*\left[V_T\right] \leq v_0 \end{cases} \tag{19}$$

Föllmer and Leukert demonstrated in Föllmer and Leukert (2000) that such problem can also be solved with the help of convex duality methods, similar to utility maximisation, as one can define *state-dependent utility function*

$$u(x, \omega) = U\left(H(\omega)\right) - U\left(\left(H(\omega) - x\right)^{+}\right).$$

And then re-write (19) in the following form

$$\begin{cases} \mathbb{E}\left[u\left(V_T, \omega\right)\right] = \max \\ \sup\limits_{P^* \in \mathcal{P}} \mathbb{E}^*\left[V_T\right] \leq v_0 \end{cases}$$

which can be solved explicitly on the complete market.

For each $z \leq E^*[H]$ there is a unique terminal wealth $\tilde{Z}$ such that

$$E\left[U\left(\tilde{Z}, .\right)\right] = \sup\left\{E\left[U(Z, .)\right] \middle| 0 \leq Z \leq H, E^*[Z] \leq z\right\}.$$

It takes the form

$$\tilde{Z} = I\left(y(z) Z_T^0(\omega), \omega\right) \wedge H(\omega)$$

where $y(z)$ is the solution of

$$E^*\left[I\left(y(z) Z_T^0(\omega), \omega\right) \wedge H(\omega)\right] = z$$

Obviously, such a reduction provides strong evidence that one can move on in the direction of Theorem 1 to elaborate on mentioned criteria and generalise them for this category of problems.

Quantile Hedging

An important case of efficient hedging is when we focus on minimising the expected size of the shortfall, or $U(X)=X$. This particular case is extremely useful for solving (17) to find the solution of CVaR optimisation problem (16).

Apart from applying similar convex duality methods Spivak and Cvitanic (1999), one can use an alternative approach, which involves the famous Neyman-Pearson lemma. According to Föllmer and Leukert (1999), it is enough to solve the equivalent problem

$$\begin{cases} \int \varphi dQ \to \max \\ \int \varphi dQ^* \le \dfrac{\tilde{V}_0}{E^*[H]}, \ \forall P^* \in P \end{cases} \tag{20}$$

where $$\frac{dQ}{dP}=\frac{H}{E[H]}, \quad \frac{dQ^*}{dP^*}=\frac{H}{E^*[H]}.$$

The solution to problems of such a type was demonstrated in Föllmer and Leukert (1999) and can be found as a perfect hedge for a modified claim $\tilde{H}=H\tilde{\varphi}$ where

$$\tilde{\varphi}=I_{\frac{dP}{dP^*}>\tilde{a}}+\tilde{\gamma}I_{\frac{dP}{dP^*}=\tilde{a}}$$

$$\tilde{a}=\inf\left(a\ge 0 \,|\, E^*\left[HI_{\frac{dP}{dP^*}>\tilde{a}}\right]\le\tilde{V}\right)$$

$$\tilde{\gamma}=\frac{\tilde{V}-E^*\left[HI_{\frac{dP}{dP^*}>\tilde{a}}\right]}{E^*\left[HI_{\frac{dP}{dP^*}=\tilde{a}}\right]} \tag{21}$$

It is easy to notice that the solution is based on finding maximal successful hedging set, which can be represented as $\left\{\dfrac{dP}{dP^*}>Const\times H\right\}$, where H is some claim. With the reasonable assumption that claim H depends on some existing asset S_T^i and using the following representation on the complete market

$$\frac{dP_T}{dP_T^*}=\exp\left\{\theta^T\tilde{W}_T-\frac{1}{2}\theta^2T\right\}=\left(S_T^i\right)^{1/\varphi^i}\times\Lambda^i$$

where $\varphi^i=\dfrac{\mu^i-r}{\|\theta\|^2}$, successful hedging set can be found in the form of

$$\left\{\left(S_T^i\right)^{\frac{1}{\varphi^i}}\times\Lambda^i>Const\times H\left(S_T^i\right)\right\}$$

which, in the case of one dimension, coincides with the solution described in Melnikov et al. (2001).

In an incomplete market case, we again add some auxiliary assets into consideration. As was demonstrated above, one can develop innovative Brownian Motion, under which, last $(k-n)$ coefficients of each row σ^i for existing assets in the "completed" volatility matrix will be equal 0. Then, using representation (6), if claim H still depends on existing assets only, it is possible to show that

$$\left\{\frac{dP_T}{dP_T^*}>a\cdot H\right\}= \ \left\{Z_{asset}^{-1}\cdot Z_{completion}^{-1}>a\cdot H(asset)\right\}$$

Consequently, it is reasonable to develop a general theory of applying Method of Market Completions to the construction of a successful hedging set. It helps reduce the Quantile Hedging problem to operations with existing assets only.

6 Conclusion

In this paper Method of Market Completions is introduced as a dual approach for operating on incomplete markets. It was demonstrated that in the case of pricing problem, this approach leads to the same solution as classical ones. As the method of market completions offers an alternative way of working with standard problems of mathematical finance in incomplete markets, it was shown how to reduce such problems to the known version in the complete market.

In line with it, alternative ways of handling market incompleteness were observed with their connection to the method of market completions and possible future developments and improvements of the presented method. Further enhancements of this method consist in finding a way for reverse-engineering parameters of the completion required utilising BSDE, partial equilibrium market condition or using another asset class like bonds or insurance contracts. On the other hand, it will also be beneficial to find a way of choosing the most suitable completion according to market conditions and investors goals.

Acknowledgements

The research was supported by the NSERC Discovery Grant RES 0043487.

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The article was submitted on 17.05.2021, reviewed on 16.07.2021, and accepted for publication on 17.08.2021. The authors read and approved the final version of the manuscript.
Статья поступила в редакцию 17.05.2021; после рецензирования 16.07.2021; принята к публикации 17.08.2021.
Авторы прочитали и одобрили окончательный вариант рукописи.

ORIGINAL PAPER

DOI: 10.26794/2308-944X-2021-9-3-94-102
JEL C13, C53, D21, D81, J31

About Some Risks Associated with Subjective Factors, and the Methodology for their Assessment

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ABSTRACT

The authors propose a methodology for assessing the risk associated with subjective factors that may affect the achievement of the final goals of business projects, including ensuring information security. Such factors may include the level of salary, the level of professionalism, and others. At the same time, we propose carrying out the risk assessment by using the fuzzy logic method, which allows us to determine the dependence of the risk on various parameters under conditions of their uncertainty. According to the authors, the proposed methodology will help avoid some incorrect management decisions in the formation of author (working) teams, which could lead to negative consequences in the further implementation of the business project. These negative consequences can be expressed in delaying the implementation period, increasing the project's cost, or even losing business due to critical information and personnel leakage. Also, this method allows you to increase the effectiveness of personnel policy in the organisation or the company. We noted that this method is applicable not only for individual enterprises but also for corporations and associations with complex network structures.

Keywords: business project; qualified leak; information security; external and internal violator; human factor; fuzzy logic; risk management

For Citation: **Kozlov, A., & Noga, N. (2021). About some risks associated with subjective factors, and the methodology for their assessment. *Review of Business and Economics Studies, 9*(3),94-102. doi:10.26794/2308-944X-2021-9-3-94-102**

ОРИГИНАЛЬНАЯ СТАТЬЯ

О некоторых рисках, связанных с субъективными факторами, и методика их оценки

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АННОТАЦИЯ

Авторами предложена методика оценки риска, связанного с субъективными факторами, которые могут оказывать влияние на достижение конечных целей бизнес-проектов, включая обеспечение информационной безопасности. В качестве таких факторов могут выступать: уровень зарплаты, уровень профессионализма и другие. При этом оценку риска мы предлагаем проводить с помощью метода нечеткой логики, что позволяет определять зависимость риска от различных параметров в условиях их неопределенности. По мнению авторов, предлагаемая методика поможет избежать некоторых неправильных управленческих решений при формировании авторских (рабочих) коллективов, которые могли бы привести к негативным последствиям при дальнейшей реализации бизнес-проекта. Эти негативные последствия могут выражаться в затягивании сроков реализации, удорожании самого проекта или даже потере бизнеса из-за утечки критически важной информации и кадров. Представленная авторами методика позволяет повысить эффективность проведения кадровой политики не только в отдельных организациях, но и в корпорациях и объединениях, имеющих сложные сетевые структуры.

Ключевые слова: бизнес-проект; квалифицированная утечка; информационная безопасность; внешний и внутренний нарушитель; человеческий фактор; нечеткая логика; управление риском

1 Introduction

In current conditions, any organisation (enterprise) starting a new business project must determine the purpose (aim) of this project, the necessary funds and resources for its implementation, and possible risks.

The document titled GOST R ISO 31000–2019 "Risk Management. Principles and guidelines" defines risk as: "the consequence of the influence of uncertainty on the achievement of aims".

This influence can lead to a deviation in achieving the aim. The deviation can be expressed as a failure of deadlines, an increase in costs, or even a complete failure of the project and, as a result — the loss of business.

The longer the project's duration, the more likely it is that its implementation's external and internal conditions will change. It means that long-term projects are a priori riskier than short-term ones.

Currently, no solid business project is complete without the use of information technology. And these technologies both help to speed up all processes and bring with them new threats and risks.

According to the InfoWatch group of companies, for the first nine months of 2020, 7.4 per cent fewer leaks were registered in the world than in the same period last year [InfoWatch, 2020]. On the contrary, in Russia, the number of leaks increased by 5.6 per cent over the same period. From January till September 2020, 9.93 billion records of personal data and payment information were leaked worldwide, of which 96.5 million were in Russia. The leaks distribution by data type we present in Table 1.

During the same period, 52.6 per cent of leaks worldwide occurred due to external influences. At the same time, there was only 21 per cent of such leaks in Russia, and more than 79 per cent of leaks occurred due to internal violations. If a little more than half of the violations of an internal nature are recognised as intentional in the world, then in Russia, there are more than 3/4 of such violations. In Russia, the share of leaks caused by employees is twice as high as in the world — more than 72 per cent. The leaks distribution by culprit we present in Table 2.

More than 40 per cent of registered leaks in Russia are in the high-tech and financial sectors — 21.9 per cent and 18.9 per cent of cases, respectively. In the world, the high — tech sector is in the first place with a share of 19.4 per cent, and healthcare is in second place — 16.4 per cent.

In Russia, the share of leaks associated with fraudulent activities is three times higher, 10.3 per cent versus 3.3 per cent. It means that violators, primarily internal ones, still have many loopholes to take advantage of information stolen from the corporate circuit for direct profit.

The main channel of leaks remains the Network (Browsers and the Cloud).

Also, in Russia, the share of leaks through paper documentation remains relatively high. Despite the rapid development of electronic document management in recent years, a significant part of the data is still stored and transmitted on paper.

The statistic shows that in 2019 internal leaks of information constituting a trade (commercial) secret occupy firmly the second place after the undisputed leader — internal leaks of personal data: 75 and 12 per cent, respectively.

At the same time, it should be borne in mind that leaks of information constituting a commercial secret are intentional in 80 per cent of cases. Leaks of personal data, on the contrary, are mostly accidental.

In the case of user data, more than half of the leaks are accidental. In the case of other types of data, most of the leaks occur due to deliberate actions. Intentional leaks count for commercial secrets

Table 1
Distribution of leaks by data type: Russia-World, January-September 2020

Type of the data	In Russia (%)	In the world (%)
Personal information	85.9	80.1
Payment (financial) information	2.0	5.6
State secret	6.7	4.7
Business secrets, know — how	5.4	9.6

Source: The authors.

(80 per cent), production secrets (88 per cent), and state secrets (85 per cent).

At the same time, internal intentional leaks have high latency. An internal violator "targeting" the theft of the employer's trade secrets is usually well aware of where the information of interest is stored, how and who controls the data transmission channels. As a result, the leak of commercial secrets is either not recorded at all or is discovered by the affected company after the fact.

Internal leaks have powerful destructive potential. The consequences of mistakes or malicious actions of personnel can manifest themselves in property or reputational losses and the suspension or liquidation of the business.

The factors influencing the actions of the internal violator are usually subjective and have a corruption component at their core [Kozlov & Noga, 2019]. When assessing the risk of implementing a business project, it is necessary to consider these factors.

2 Subjective Risk Factors

What motivates the internal violator? The main reasons are greed and negligence. The self-serving and psychological motives for violations are almost the same as those of corruption [Vannovskaya, 2013]:

- The employee's opinion that his work is undeservedly undervalued
- A significant difference in different categories of employee's wage
- High staff turnover, the presence of "temporary workers", including among managers
- Lack of individual employee interest in achieving the project aim, personal dissatisfaction
- Low qualification of the employee, his inability to work on equal terms
- Company tolerance to minor violations
- The presence of double standards in the organisation when a certain category of employees (managers) is allowed to violate the established rules. The employee believes that this is cheating him, and he also has the right to cheat
- Excessive bureaucracy and insufficient control, when it is easier to circumvent the rules than to comply with them.

Risk Parameters

Consider the above factors as some parameters that affect the value of risk in the organisation. The level of material satisfaction consists of wage and household comfort. Several parameters can define this level:

- The value of the deviation of the average wage in the team from the average salary in the industry (region)
- The ratio of the average wage in the team to the average wage in the industry
- The spread of employees' wages (how much they differ in the team), its dispersion.

The authors propose representing wage dispersion as the standard deviation from the average value, well described by the variance of a discrete random variable — wage:

$$D(z) = M(z - M(z))^2, \quad (1)$$

where D (z) is the wage dispersion (the average of the square of the deviation of the wage from the average level), z — is a random variable — the employee's wage, M (z) — is the average wage in the team.

According to the author's opinion, another important parameter that affects the organisation's risk is the professional level of employees. This level can be represented as the ratio of the average employees work experience in this area (in this direction) to the average life cycle of products (products) produced by this company.

Under the product life cycle, we will understand the time required for its development, testing, organisation of production, production cycle, the period

Table 2
Distribution of leaks by violators: Russia — World, January — September 2020

Leak's violator	In Russia (%)	In the world (%)
Head manager	5.0	2.6
System Administrators	0.0	0.1
Unprivileged employees	72.1	36.5
Former employees	2.0	0.9
Contractors	1.0	2.2
External attackers	29.1	57.7

Source: The authors.

of implementation and operation, during which its technical support is carried out.

Thus, the professional level can be represented in the following form:

$$P = \frac{\sum_{i=1}^{n} S_i}{nG}, \qquad (2)$$

where P — is the level of professionalism, S_i — is the employee's work experience in this field, n is the number of employees in the team, and G — is the average life cycle of the products produced.

At the same time, it is worth noting that the more technologically complex products usually have longer life cycles. So, for example, it can take ten or more years to develop an entirely new computer processor based on new architectural principles or create a new aircraft. And to launch it into production with the solution of numerous organisational issues may take as many more years.

Also, the following parameters can be attributed to subjective risk factors:

- The level of the employee's interest in the results of the work, defined as the time of work on this project (usually, if the employee is interested in the result, then he tries, all other things being equal, to stay in the team until the final result is obtained)
- The level of comfort in the team can be determined by the parameter — the lifetime of the stable core of the team (the stable core of the team). If it were not comfortable to work in this team, then employees would try to leave it, and there would be a significant turnover of personnel
- The level of commitment to the company's goals is a parameter similar to the previous one, only, in this case, it refers to a large company and is instead an individual parameter for a particular employee, i.e., the longer the employee's work experience in this company, the higher the level of commitment
- The level of compliance of the vector of decisions made with the company's goals and their impact on other team members. This parameter rather refers to top managers who make decisions or influence the adoption of certain managerial decisions. For example, a company produces aeroplanes or cars but has faced some financial issues. The financial manager, first of all, should reduce expenses. But, if at the same time to reduce the division of designers — designers, so in the future will not be created new aircraft or cars and the company will not be competitive, and may even lose business.

The human (subjective) factor is essential when assessing the risk for any enterprise and high-tech companies working with new technologies — especially. Underestimating this can negatively affect the performance of the enterprise (company).

Therefore, it is necessary to maintain a decent wage level, strive to maintain and, if possible, create a healthy climate in the team based on the professionalism of its employees and provide opportunities for career and material growth to reduce the risk associated with subjective factors.

But how to assess this risk depend on the listed parameters, which is not clearly expressed?

In this case, the authors suggest using the fuzzy logic method for its evaluation and using the MATLAB Fuzzy Logic Toolbox package for its implementation [Matlab, 2019].

3 Risk Assessment

In assessing the risk according to the proposed methodology, it is possible to build the depend-

Table 3
Wage (salary) level

N	Wage level	Possible actions of employees	Relation to the average wage in the sector	The boundaries of the term "Wage level"
1	Low	Employees ' desire to find another job or to sell secrets	0.1–0.75	0.1–0.4
2	Middle	Stable job, but getting a better offer, leave your job	0.75–1.50	0.4–0.6
3	**High**	**The desire to maintain this level**	**More 1.50**	**0.6–1.0**

Source: The authors.

Table 4
Wage dispersion level

N	Wage dispersion level	Possible actions of employees	The boundaries of the term "Wage dispersion level"
1	High	Employees' desire to find another job or to sell secrets	0.7–1.0
2	Middle	Stable job, but getting a better offer, leave your job	0.3–0.8
3	**Low**	**The desire to maintain this level**	**0.1–0.4**

Source: The authors.

ence of the risk on all of the above parameters and some other specific ones to specific enterprises. But in this case, the description and calculation will be pretty time-consuming. A variant of the risk assessment with dependence on five parameters is given in the paper [Kozlov & Noga, 2020].

For simplicity and clarity, we will evaluate the dependence of risk on three parameters, which, according to the authors, are one of the main parameters that influence subjective risk factors.

In the proposed example, these parameters will be:

- The wage level *U(z)* — the ratio of the average salary in the team *M (z)* to the average wage in the industry *M*
- The wage dispersion level in the team *D(z)*
- The professional level of the employees *P*.

In this case, the risk R can be represented as a function of these parameters.

$$R = R(U(z), D(z), P) \quad (3)$$

The fuzzy logic method involves working with linguistic variables. The correspondence of linguistic variables to the above parameters we show in Tables 3–5. We will consider all variables normalised with values in the range from 0 to 1.

If the company's wage level is significantly lower than the average in the industry, then the company will inevitably face problems with recruiting qualified specialists. The exception may be cases of temporary difficulties with the prospect of overcoming them in the future.

Significant dispersion of the employees' wage in the team can also cause many negative cases, such as envy and betrayal of the company's interests based on "underestimating" the personal employee's contribution.

Moral and material dissatisfaction can push an employee (employees) to find a new job with better working conditions and simply to sell the technical and technological secrets of the company.

The professional level also has a significant impact on the risk assessment. The lower this level, the more likely it is to make mistakes that can lead to the failure of the project deadlines, its price rise, or even to the inability to achieve the aim.

In addition, less professional employees are more prone to overestimating their importance and sometimes do not listen to the opinions of more experi-

Table 5
Professional level

N	Professional level	Possible consequences	The boundaries of the term "Professional level"
1	Low	Possible adoption of technically incorrect decisions	0.1–0.3
2	Middle	Increasing the project implementation time, reducing its quality due to insufficient experience	0.3–0.8
3	High	There may be minor deviations in the implementation time	More 0.8

Source: The authors.

Table 6
Output variable Risk (yR)

	Risk level	The boundaries of the term "Risk level"
1	Insignificant	0.0–0.20
2	Acceptable	0.16–0.50
3	High	0.45–1.00

Source: The authors.

enced employees. And if they are also top managers, head of the team or company divisions with the vote right, then the consequences can be very harmful. The linguistic variable with the corresponding professional level we show in Table 5.

Finally, an approximate risk estimation algorithm based on the provisions of fuzzy logic and fuzzy set theory, considering the uncertainties that arise in any organisation, can be implemented using the as mentioned above MATLAB Fuzzy Logic Toolbox package. When using the production rules of fuzzy logic, we reproduce the output mechanism taking into account the three input variables. Such variables for assessing the risk associated with subjective (human) factors in our example, as already mentioned above, are:

- wage level
- wage dispersion level
- employees professional level.

Each of the listed input variables, as indicated above, is evaluated on its own scale. Next, these input variables are passed to the Fuzzy Logic Toolbox, and the output is the value of the output variable — risk.

As a visual example, consider a simplified risk calculation in the Fuzzy Logic Toolbox with three input variables: wage level — x_Z, wage dispersion level — x_D, and professional level — x_P.

There is the variable risk — y_R (Risk). That is, now equation (3) has the following form

$$y_R = R\,(x_Z, x_D, x_P). \qquad (4)$$

We apply the Mamdani model and assume that the membership functions of the three variables have a trapezoidal form. The risk membership function has the shape of a Gaussian curve. The ranges of changes in terms specified in Tables 3–5, respectively, are used to evaluate the input variables. For the output variable y_R, we use three terms with the measurement range specified in Table 6.

Further, to form a fuzzy knowledge base, we introduce production rules, partially presented in Table 7.

A graphical representation of the Mamdani knowledge base in the Fuzzy Logic Toolbox rules editor we show in Figure 1.

After defuzzification, you can get a specific value of the output risk parameter for specific values of the input variables and compare it with the acceptable value. The presented package allows you to visualise the dependence of the risk on the input parameters (4).

According to the given input parameters, it is possible to build three three-dimensional graphs. At the same time, on each of them, you can see the dependence of the risk for two parameters with fixed values of the third. To determine the optimal values of the parameters with an acceptable

Table 7
Fuzzy knowledge base, production rules

N	Wage level	Wage dispersion level	Professional level	Risk level
1	Low	High	Low	High
2	Low	High	Middle	High
3	Low	High	High	High
4	Low	Middle	Low	High
…	…	…	…	…
24	High	Middle	High	Insignificant
25	High	Low	Low	Acceptable
26	High	Low	Middle	Insignificant
27	High	Low	High	Insignificant

Source: The authors.

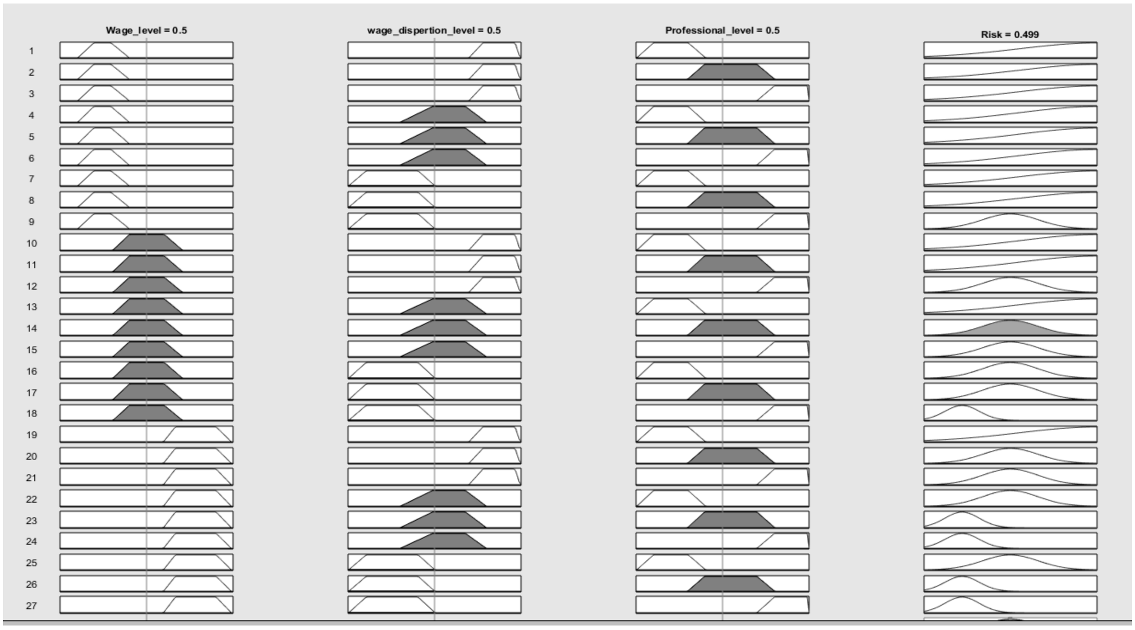

Figure 1. Mamdani's fuzzy knowledge base in the Rules Editor.

Source: The authors.

risk value, you need to do some work, varying the values of the input parameters. Naturally, this is only possible within the limits of the restrictions imposed on these values, which are available to implement real projects.

Figure 2 shows a visualisation of the risk dependence at the wage dispersion level and the wage level. This relationship indicates that the smaller the wage dispersion level in the team and the higher the overall wage level, the lower risks associated with the manifestation of subjective (human) factors.

You can also visualise the risk dependence at the professional level and the wage level (Figure 3), the professional level and the wage dispersion level (Figure 4). In this example, to simplify the presentation of the basic principles of the proposed method for assessing the dependence of risk on subjective factors, each linguistic variable corresponds to only three intervals of values. In fact, you may need more of them to get more accurate results. And there may be more variables themselves. For example, to assess the level of wage, it may be necessary to compare it with the level of living in a given country and the wage level of a specialist in a given professional field in other countries.

It is necessary to understand that the further use of new values and new variables increases the number of production rules and complicates their writing. It, in turn, may lead to the need to attract additional experts.

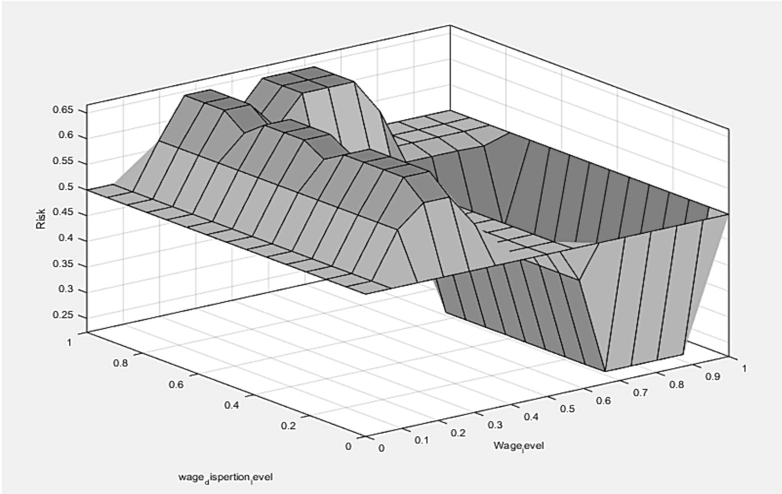

Figure 2. Visualisation of the risk dependence at the wage dispersion level (x_D) and the wage level (x_Z)

Source: The authors.

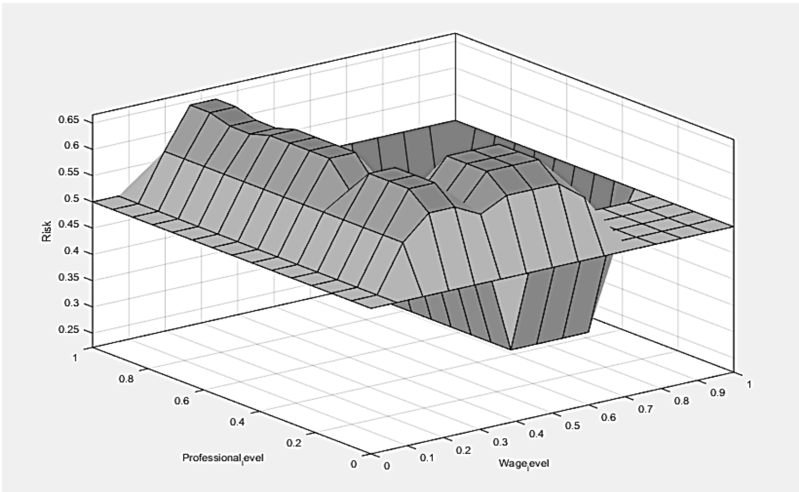

Figure 3. Visualisation of the risk dependence at the professional level (x_P) and the wage level (x_Z).

Source: The authors.

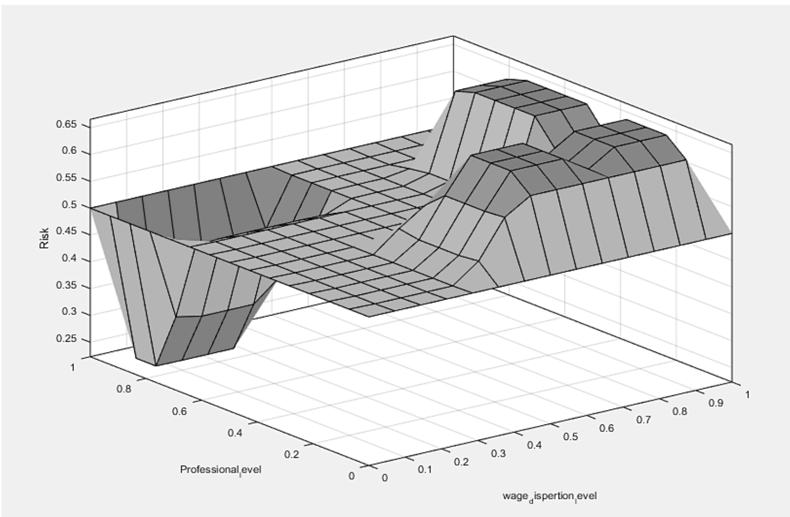

Figure 4. Visualisation of the risk dependence at the professional level (x_P) and the wage dispersion level (x_D).

Source: The authors.

4 Conclusion

The proposed method allows us to assess the dependence of risk on subjective factors that are difficult to describe mathematically strictly. The article provides an example of evaluating the dependence of risk at the wage level, the wage dispersion (spread) level and the professional employees level. This method allows us to assess the dependence of risk on other subjective parameters, both those given in this paper and those that may be specific only for specific enterprises or company.

Using the above methodology, in conditions of great uncertainty and non-obvious mutual influence of parameters at different stages of the life cycle of various business projects, it becomes possible:

1. Determine the impact of various subjective risk factors on the level of a particular business project implementing risk

2. Assess the level of risk, both at the moment and at various stages of the business project life cycle

3. Optimise the personnel policy of the enterprise (organisation), which reduces the risk of leakage of high-tech (know-how) information, as well as the leakage of “brains”, to stabilise the staff

4. Develop recommendations for the formation of a healthy atmosphere in the team, which will allow you to optimally solve the tasks set to achieve the aims of the business projects

5. Avoid erroneous management decisions, especially those related to the company or staff “optimisation”.

The proposed methodology can be used in individual enterprise and organisations with a complex network structure. For example, there may be a company with a large number of branches. If so, it is necessary to compare the wage level not within the team but between branches.

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The article was submitted on 17.05.2021, reviewed on 16.07.2021, and accepted for publication on 17.08.2021.
The authors read and approved the final version of the manuscript.
Статья поступила в редакцию 17.05.2021; после рецензирования 16.07.2021; принята к публикации 17.08.2021.
Авторы прочитали и одобрили окончательный вариант рукописи.